

Dimensional Analysis

A ship of hull length 140m is to travel at 7.6 m/s in salt water.

- a) At what speed should a 1:30 scale model of the ship be towed in salt water? (15 marks)
- b) Explain the dimensionless number you used to describe similitude between the model and the prototype, and comment on the validity of the results. (15 marks)

a) As ship motion is a surface phenomenon dominated by gravity forces we can use the Froude number for similitude.

Froude number in ratio form: $Fr_r = \frac{V_r}{\sqrt{y_r}} = 1 : V_r = \sqrt{y_r}$

That is the ratio of kinematically similar velocities between model and prototype is equal to the square root of their length ratio, which is the model scale:

$$V_r = \frac{V_m}{V_p} = \sqrt{y_r} = \sqrt{1/30} = 0.1826$$

$$V_m = 0.1826V_p = 0.1826(7.6) = 1.388 \text{ m/s}$$

b) We recognise the need for both Reynolds and Froude number similitude (for both friction drag and wave drag) in the testing of hydrodynamic models, however there is a necessary inconsistency between them that can only be resolved by use of a model fluid of controlled kinematic viscosity. This is not always possible. It is generally accepted however that ship speeds are dominated by gravity in the bow-wave and thus Froude similitude is an acceptable first approximation in this case..

A drag force of 43 N was measured on a 2.44 m model ship towed at 1.98 m/s in fresh water (density 1000 kg/m³).

- a) (20 Marks) What force would be required to drive a 39.04 m prototype ship in salt water (density 1025 kg/m³) at a kinematically similar velocity?
- b) (10 Marks) What would be the velocity of the prototype ship in those conditions? (Exam 2009) Assignment 4 2010

a) Again we use the Froude number, with some reservations. In addition we can use the Euler number to work with force ratios:

Froude number ratio: $Fr_r = \frac{V_r}{\sqrt{y_r}} = 1 : V_r = \sqrt{y_r}$

Euler number ratio: $Eu_r = \frac{P_r}{\rho_r V_r^2} = 1 \quad : \quad P_r = \rho_r V_r^2$

Force ratio: $F_r = P_r A_r = P_r y_r^2 = \rho_r V_r^2 y_r^2 = \rho_r y_r^3$

The force ratio, assuming similar force distributions arising from constant pressure ratios on corresponding areas of the model and prototype, is proportional to the cube of the model scale. We can use this to equate forces on a model and prototype at kinematically similar velocities:

$$F_r = \frac{F_m}{F_p} = \rho_r y_r^3 = \frac{1000}{1025} \left(\frac{2.44}{39.04} \right)^3 = 238.2 \times 10^{-6}$$

$$F_p = F_m / 238.2 \times 10^{-6} = 43 / 238.2 \times 10^{-6} = 180.5 \times 10^3 \text{ N}$$

b) The kinematically similar prototype velocity at which this force applies can be determined from the Froude velocity ratio:

$$V_r = \frac{V_m}{V_p} = \sqrt{y_r} \quad : \quad V_p = \frac{V_m}{\sqrt{y_r}} = \frac{1.98}{\sqrt{2.44/39.04}} = 7.92 \text{ m/s}$$

A 1:225 scale model of a reservoir is drained in 4.25 minutes by opening a sluice gate. How long will it take to empty the full scale prototype?

The system's behaviour is driven by gravitational forces acting on free surfaces, so the Froude number governs similitude:

$$Fr_r = \frac{V}{\sqrt{gy}} \Bigg]_{m/p} = \frac{V}{\sqrt{gy}} \Bigg]_r = \frac{V_r}{\sqrt{g_r y_r}} = 1$$

Given the same gravitation in model and prototype we can identify the velocity ratio from model to prototype as

$$V_r = \sqrt{y_r} \text{ where } y_r = L_r \text{ is the length scale, ie the ratio of lengths.}$$

A combination of length ratio and velocity ratio determines the time ratio, ie the ratio of times to characteristic points such as complete emptying:

$$L_r = V_r t_r = \sqrt{y_r} t_r = \sqrt{L_r} t_r$$

Thus the time ratio is

$$t_r = t_m/t_p = L_r/\sqrt{L_r} = \sqrt{L_r}$$

Thus the time the prototype will take to empty is given by

$$t_p = t_m/t_r = t_m/\sqrt{L_r} = 4.25/\sqrt{1/225} = 63.75 \text{ minutes}$$

A 1:4 scale model of a torpedo is tested in a salt water towing tank at a velocity of 24.4 m/s and thrust of 12.2 kN. What velocity can we expect the prototype to attain with an appropriately scaled thrust? What thrust should the prototype exert to attain that velocity?

Forces in this (submerged) system are dominated by viscous friction and momentum, hence the Reynolds number governs similitude:

$$Re_r = \frac{Re_m}{Re_p} = \frac{\rho_r D_r V_r}{\mu_r} = 1$$

The fluid density is the same for model and prototype, so $\rho_r = \mu_r = 1$. The velocity ratio is thus inverse to the diameter ratio, which is the ratio of model to prototype diameters or more simply the length scale:

$$V_r = \frac{V_m}{V_p} = \frac{1}{D_r}$$

The prototype velocity is then

$$V_p = V_m D_r = V_m \frac{1}{4} = 24.4/4 = 6.1 \text{ m/s}$$

Forces such as thrust and drag can be evaluated as model:prototype ratios through the use of the Euler number:

$$Eu_r = \frac{P_r}{\rho_r V_r^2} = 1$$

A force ratio can be evaluated by pressure and area ratios:

$$F_r = P_r A_r = \rho_r V_r^2 A_r = \rho_r V_r^2 L_r^2$$

With $\rho_r = 1$ and V_r determined by Reynolds similitude as before we have

$$F_r = \frac{L_r^2}{L_r^2} = 1$$

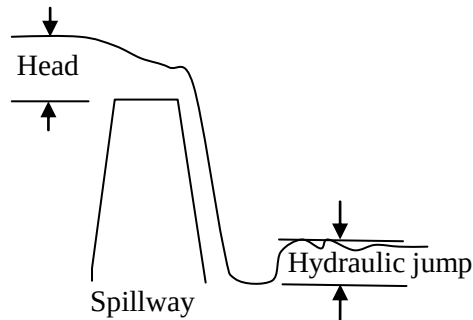
This means the model and prototype must apply the same thrust for kinematically similar velocities. If the 1:4 scale model generates 12.2 kN of thrust to reach 24.4 m/s, the prototype must produce the same thrust to reach 6.1 m/s.

A spillway model is to be built to a scale of 1:25 across a laboratory flume which is 0.61 m wide. The prototype will be 11.4 m high and the maximum actual head expected is 1.52 m.

a) What height of model and what head of model should be used?

b) If the flow over the model head is 0.02 m³/s, what flow per m can be expected in the prototype under similitude conditions? (consider the dominant force on the free surface)

- c) If the model shows a hydraulic jump of 26mm, how high is the jump in the prototype?
- d) If the energy dissipated in the model hydraulic jump is 112 W, what energy dissipation will occur in the prototype?



- (a) The model height and head are determined by the prototype height and the scale of 1:25:

$$L_m = L_p/25 = 11.4/25 = 0.456 \text{ m}$$

$$H_m = H_p/25 = 1.52/25 = 0.061 \text{ m}$$

- (b) The dominant force on the free surface is gravity, so we need Froude number similitude. Using the Froude number we need to find an expression for flowrate:

$$F = V/(gL)^{1/2}$$

Froude similitude implies the model and prototype Froude numbers are the same:

$$F_m/F_p = F_r \text{ (r for ratio)} = V_r/(g_r L_r)^{1/2} = V_r/(L_r)^{1/2} \text{ since } g_r = 1$$

(model and prototype in the same g field)

But $F_r = 1$

Thus $V_r = L_r^{1/2}$

Ie the velocity ratio is to the square root of the length ratio or scale.

The velocity ratio allows us to determine a flow ratio:

As $Q = VA$

$$Q_r = V_r A_r$$

$$= L_r^{1/2} L_r^2 = L_r^{5/2}$$

Thus the discharge ratio is to the 5/2 power of the length ratio.

So $Q_m/Q_p = (L_m/L_p)^{5/2}$

$$Q_p = Q_m/(L_m/L_p)^{5/2} = 0.02/(1/25)^{5/2} = 62.5 \text{ m}^3/\text{s}$$

- (c) Again, the height or length of any aspect of the model and prototype are related by the length scale:

$$H_p = H_m/L_r = 26/(1/25) = 650 \text{ mm}$$

- (d) Here a Froude-similitude form of the expression for a power ratio is needed.

$$[\text{Power}]_r = F_r V_r = M_r L_r^2 T_r^{-3} = \rho_r L_r^3 L_r^2 T_r^{-3}$$

Water is used in both model and prototype so the density ratio is 1

The time ratio can be worked out using Froude ratio:

$$F_r = 1 = V_r/L_r^{1/2} : V_r = L_r/T_r = L_r^{1/2}$$

$$T_r = L_r/L_r^{1/2} = L_r^{1/2}$$

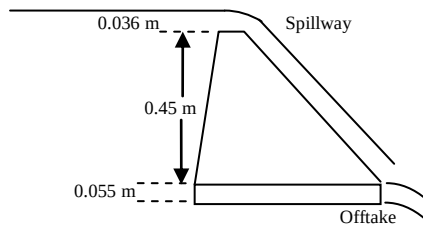
$$\text{So } [\text{Power}]_r = L_r^3 L_r^2 L_r^{-3/2} = L_r^{7/2}$$

$$\text{And } P_p = P_m/L_r^{7/2} = 112/(1/25)^{7/2} = 8.75 \times 10^6 \text{ W} = 8.75 \text{ MW}$$

A model dam at a scale of 1:25 has a spillway of width 30 cm on top of a wall 45 cm high at the spillway. The model dam also has a low level offtake modelled as a single short horizontal 5.5 cm diameter pipe through the base of the wall.

When the water surface level behind the model dam is 3.6 cm above the spillway, the flowrates from the spillway and fully open low level offtake are measured at 3.7 litres per second (spillway) and 4.2 litres per second (offtake).

- What dimensionless number dominates both the spillway and the low level offtake? Explain in terms of the dominant force governing flow in the spillway and the pipe.
- In the full scale dam, flowing under geometrically similar conditions, what flow rates can be expected from the spillway and the low-level pipe (Use the dimensionless number in a) to derive a discharge ratio Q_r between model and prototype)
- Check the prototype flowrates derived in b) by flow measurement models based on Bernoulli. For the spillway flow use the sharp-crested weir equation with a correction for effective weir length. For the offtake flow use the orifice equation with a discharge coefficient C_d of 0.6. Why is the second one less accurate?
- The flow from the low level offtake is directed onto a spring-loaded vane, which indicates a reaction force of 5 N. Use the flow velocity from the model orifice equation to derive a power rating for the offtake outflow. Assuming an 85% efficiency, what power could be delivered by a turbine placed in the full scale low level outflow in this flow state?



Draw a diagram!!!

a) The key dimensionless number is the Froude number, representing systems dominated by gravity acting on a free surface. This applies to the pipe flow as well, since it is short and unlikely to generate significant friction effects. The use of the orifice equation in this situation shows that simple pipe flow can in some cases be examined approximately as an orifice without considering friction on the submerged length in any detail.

b) The Froude Number $Fr = V/\sqrt{(gy)}$

In similar flow situations (geometrically similar depths) a model and prototype will have similar Froude numbers. A useful shortcut is to divide the model Fr_m by the prototype Fr_p to obtain a Froude ratio. Since model and prototype are tested in the same gravity, g cancels and we are left with

$$Fr_r = V_r/y_r^{1/2} = 1$$

Here the r subscript indicates a ratio of model to prototype. y_r is the model scale $1/25$, since the height of the model y_m and the height of the prototype y_p are in that ratio.

$$y_r = 1/25$$

$$\text{Thus } V_r = y_r^{1/2} = 1/25^{1/2} = 1/5$$

ie the model velocity at key points is one fifth of the prototype velocity. This suggests how we can work out a discharge ratio:

Since $Q = VA$ (continuity)

$$\text{So } Q_r = V_r A_r$$

Where A_r is the Area ratio between model and prototype. This is related to the length ratio or scale, since for any pair of characteristic lengths that are scaled by y_r , their product or area will scale by y_r^2 :

$$A_r = y_r^2 = 1/25^2 = 1/625$$

$$\text{Thus } Q_r = V_r A_r = y_r^{1/2} y_r^2 = y_r^{2.5} = 1/25^{2.5} = 1/3125$$

This is the ratio of model to prototype flowrate: $Q_m/Q_p = 1/3125$

The prototype flowrate is thus $Q_p = 3125 Q_m$

We have two flowrates, the spillway and the offtake:

Spillway: $Q_m = 0.0037 \text{ m}^3/\text{s}$ **$Q_p = 3125(0.0037) = 11.6 \text{ m}^3/\text{s}$**

Offtake: $Q_m = 0.0042 \text{ m}^3/\text{s}$ **$Q_p = 3125(0.0042) = 13.1 \text{ m}^3/\text{s}$**

c) The full scale dam has the following scaled dimensions:

$$\text{Wall height} = 25 (0.45) = 11.25\text{m}$$

$$\text{Pipe diameter} = 25 (0.055) = 1.375\text{m}$$

$$\text{“Head” } H = 25 (0.036) = 0.9 \text{ m}$$

$$\text{Spillway length} = 25 (0.3) = 7.5 \text{ m}$$

For the spillway, the sharp-crested weir equation:

$$Q = 1.83L'H^{3/2}$$

$$\text{Where } L' = L - 0.2H = 7.5 - 0.2(0.9) = 7.32 \text{ m}$$

$$\text{Thus spillway } \mathbf{Q = 1.83(7.32)0.9^{3/2} = 11.5 \text{ m}^3/\text{s}}$$
 Excellent Match to Dimensionless Theory

For the offtake, the orifice equation:

$$Q = C_d A \sqrt{2gH}$$

We are told $C_d = 0.6$

$$A = \pi D^2/4 = \pi 1.375^2/4 = 1.485 \text{ m}^2$$

$$H = \text{total head behind the pipe flow} = \text{water height} + \text{wall height} + \frac{1}{2} \text{ pipe depth} = 0.9 + 11.25 + 0.688$$

ie $H = 12.84 \text{ m}$

$$\text{Thus offtake } Q = 0.6(1.485)(19.6(12.84))^{1/2} = \mathbf{14.1 \text{ m}^3/\text{s}}$$

The inaccuracy here is because the orifice equation is not an accurate model of the offtake pipe. A friction factor model would be more accurate. Reynolds number partly controls the offtake, but by how much?

d) The power of the outflow is a function of the reaction force and its velocity:

$$\text{Power} = FV$$

$$\text{We can work out the flow velocity from the orifice equation: } V = \sqrt{2gH}$$

Note that velocity is not affected by the contraction factor of $C_d = 0.6$, only the area and hence discharge are affected. Here H is the total head behind the model pipe: $H = 0.036 + 0.45 + 0.055/2 = 0.51 \text{ m}$

$$\text{Thus } V = \sqrt{2gH} = \sqrt{19.6(0.51)} = 3.16 \text{ m/s}$$

$$\mathbf{\underline{\text{And Power} = FV = 5(3.16) = 15.8 \text{ Watts}}}$$

We can scale up to the prototype using a power ratio:

Since $\text{Power} = FV$

So $\text{Power}_r = F_r V_r$

Be careful to distinguish the Force ratio F_r from the Froude number Fr .

Force is a product of Pressure and Area. We can define force, pressure and area ratios:

$$F_r = P_r A_r$$

A dynamic pressure ratio can be found from the Euler number:

$$Eu = P/\rho V^2$$

For similarity, the Euler ratio is unity and the density (same fluid in model and prototype tests) cancels:

$$Eu_r = P_r/V_r^2 = 1$$

Thus $P_r = V_r^2$

Ie dynamic pressures scale in proportion to squared velocities.

Reassembling, $\text{Power}_r = F_r V_r = P_r A_r V_r = V_r^2 A_r V_r$

We already have (from b) that $V_r = y_r^{1/2}$ and $A_r = y_r^2$

Thus $\text{Power}_r = V_r^2 A_r V_r = y_r^2 y_r^2 y_r^{1/2} = y_r^{3.5} = 1/25^{3.5} = 1/78125$

A 25x prototype expresses fluid energy at a rate 78125x the model.

Thus assuming 85% efficiency, $\text{Power}_p = 0.85(78125)\text{Power}_m = 0.85(78125)(15.8) = \underline{\underline{1.0 \text{ MW}}}$

This is an order of magnitude estimate at best. You may find answers in the range +/- 0.2 MW

Pipe Losses

Determine the velocity at which laminar flow transitions to turbulent for (a) medium fuel oil at 15.6°C (kinematic viscosity $4.41 \times 10^{-6} \text{ m}^2/\text{s}$) flowing through a 152.4 mm pipe, and (b) water at 15.6°C ($\nu = 1.13 \times 10^{-6} \text{ m}^2/\text{s}$) flowing in the same pipe.

For laminar flow, the critical Reynold's number is $Re = 2,000$:

$$Re = VD/\nu = 2000$$

$$(a) \quad V = 2000\nu/D = 2000(4.41 \times 10^{-6})/0.1524 = 0.0579 \text{ m/s}$$

$$(b) \quad V = 2000\nu/D = 2000(1.13 \times 10^{-6})/0.1524 = 0.0148 \text{ m/s}$$

Determine the type of flow occurring in a 305 mm pipe when (a) water at 15.6°C flows at 1.067 m/s and (b) heavy fuel oil at 15.6°C ($\nu = 205 \times 10^{-6} \text{ m}^2/\text{s}$) flows at the same velocity.

We need to calculate the Reynold's number and determine if it is less than or greater than the critical value of 2000:

- (a) $Re = VD/\nu = 1.067(0.305)/1.13 \times 10^{-6} = 287996 > 2000$: The flow is turbulent
(b) $Re = VD/\nu = 1.067(0.305)/205 \times 10^{-6} = 1587 < 2000$: The flow is laminar

Determine a theoretical diameter for a pipeline which is to convey $0.057 \text{ m}^3\text{s}^{-1}$ of oil a horizontal distance of 300 m, if the head loss is not to exceed 15.9 metres. At the operating temperature the specific gravity of the oil is 0.9 and the viscosity is 1.43 Pa.s. Assume laminar flow, and check with your result that this is the case. What power output is required from a pump to drive this flow?

Oil of absolute viscosity 0.1 Pa.s and relative density 0.850 flows through 3048 m of 305 mm cast iron pipe at a rate of $44.4 \times 10^{-3} \text{ m}^3/\text{s}$. What is the head loss in the pipe?

We need to determine the Reynolds number to decide the type of flow and hence the friction factor we should use:

We are not told the velocity, but can determine it from flow rate and diameter:

$$V = Q/A = 44.4 \times 10^{-3} / (\pi 0.305^2 / 4) = 0.608 \text{ m/s}$$

Then $Re = VD\rho/\mu = 0.608(0.305)850/0.1 = 1575 < 2000$ Laminar.

This gives a friction factor of

$$\lambda = 64/Re = 64/1575 = 0.041$$

And a headloss of

$$h_f = \lambda LV^2/2gD = 0.041(3048)0.608^2/(2(9.8)0.305) = 7.65 \text{ m}$$

Determine the headloss in 10 m of 5 mm glass tubing (roughness height effectively zero) (a) when water at 43.3°C ($\nu = 0.620 \times 10^{-6} \text{ m}^2/\text{s}$) flows at 1 m/s and (b) when carbon tetrachloride at the same temperature ($\nu = 0.465 \times 10^{-6} \text{ m}^2/\text{s}$) flows at the same velocity.

We need to work out friction factors and headlosses by calculating Reynolds numbers:

(a) $Re = VD/\nu = 1(0.005)/0.620 \times 10^{-6} = 8065$

This is in the transitional range $2000 < Re < 1000$ so the Blasius formula is used to calculate a friction factor:

$$\lambda = 0.3164/Re^{0.25} = 0.3164/8065^{0.25} = 0.033$$

The headloss from the Darcy-Weisbach formula is then

$$h_f = \lambda LV^2/2gD = 0.033(10)1/(2(9.8)0.005) = 3.40 \text{ m}$$

$$(a) \text{ } Re = VD/\nu = 1(0.005)/0.465 \times 10^{-6} = 10752$$

This is in the fully turbulent range so the Colebrook White formula is needed:

Guess a value of $\lambda = 0.02$

$$\text{Calculate } \lambda = \left[\frac{1}{-2 \log \left[\frac{\varepsilon}{3.7D} + \frac{2.51}{R\sqrt{\lambda}} \right]} \right]^2 = 0.032 \text{ with } \varepsilon = 0$$

Put that answer back into the formula and recalculate. This is easy to do with even a cheap modern graphical calculator: just review the formula and substitute “Ans” (for the previous answer) into the value of λ :

Second iteration: $\lambda = 0.030$

Third iteration: $\lambda = 0.030$: the formula converges on 0.030.

The headloss from the Darcy Weisbach formula is then

$$h_f = \lambda LV^2/2gD = 0.030(10)1/(2(9.8)0.005) = 3.09 \text{ m}$$

A 250 mm cast iron pipe is to carry a minimum flow of 700 l/s from a reservoir to a treatment pond 500 m away. The reservoir water level is 35 m higher than the pond surface.

- a) Is there any need for additional pump head to ensure the minimum flow using this diameter of pipe?
- b) What size pipe will carry the required flow at exactly the available head? (Exam 2011)

- a) The Darcy-Weisbach equation controls head loss in pipes:

$$h_f = fLV^2/2gD$$

$$V = Q/A$$

$$A = \pi D^2/4 = \pi(0.25)^2/4 = 0.049 \text{ m}^2$$

$$V = 0.7/0.049 = 14.3 \text{ m/s}$$

Assume $f = 0.02$ to begin with.

$$\text{Then } hf = fLV^2/2gD = 0.02(500)14.3^2/(19.62(0.25)) = 417 \text{ m}$$

Without going any further to refine the estimate, we can see that headlosses are far in excess of available fall, so pumping will be required for this pipe size.

- b) With the diameter to be specified and hence velocity unknown, but with the headloss specified, the D-W equation becomes:

$$hf = fLV^2/2gD = 35 \text{ m}$$

$$\text{But } V^2 = (Q/A)^2 = (0.7/(\pi D^2/4))^2 = 0.794/D^4$$

Assuming again a value of $f = 0.02$ we have

$$hf = 0.02(500)(0.794/D^4)/2gD = 0.405/D^5 = 35 \text{ m}$$

$$\text{This solves as } D = \sqrt[5]{(0.405/35)} = 0.410 \text{ m}$$

$$\text{For which } V = \sqrt{(0.794/D^4)} = 5.3 \text{ m/s}$$

Calculate a Reynolds number to check the friction factor:

$$Re = VDr/\mu = 5.3(0.41)1000/0.001 = 2.2 \times 10^6$$

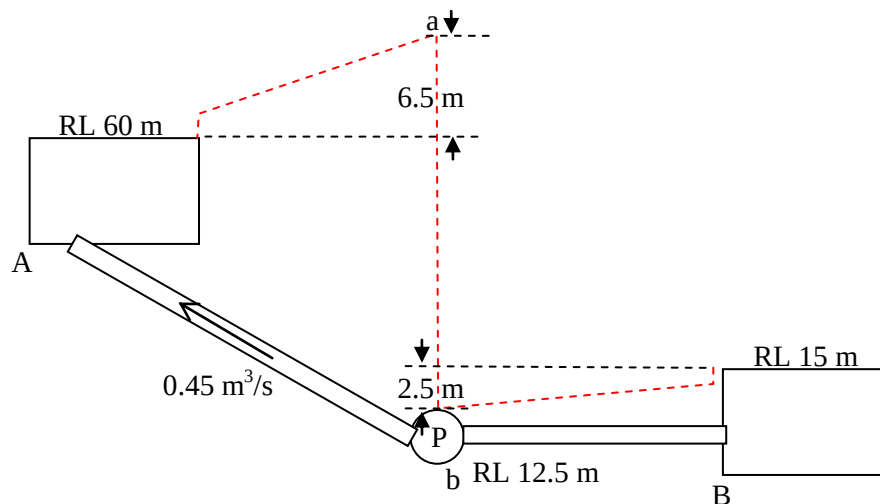
For cast iron pipe, roughness 0.24 mm, $e = 0.24/392 = 0.0006$

The Moody diagram shows a f value very close to 0.02. Solution is acceptable and a standard size is nearby to choose from:

$$D = 0.410 \sim 400 \text{ mm}$$

The two water reservoirs in a large town are connected to a pumping station. The water level in primary reservoir A is 80 m above sea level, while that of secondary reservoir B is at 10 m. The main pump in the station is at elevation 9.5 m. The pump must transfer $1.7 \text{ m}^3/\text{s}$ from the secondary to the primary reservoir, at which rate the known head losses are 3.1 m from reservoir B to the pump and 9.0 m from the pump to reservoir A. How much fluid power must the pump add to the system? Plot the energy line of the system.

Again let's start with a diagram, not necessarily to scale:



The dotted line is a sketch of the total energy line or TEL of the system. Minor entry and exit losses appear as small steps at the two reservoirs A and B. Pipe losses appear as sloping sections of the TEL. The pump's energy input appears as a major step upwards between points b and a. The size of this step, in metres of head, can be extracted from the diagram: Working from left to right, adding and subtracting the known and unknown values of head, we have:

$$60 + 6.5 - \text{pump head} + 2.5 = 15$$

So $\text{pump head} = 60 + 6.5 + 2.5 - 15 = 54 \text{ m}$

From p72 (study guide) we can convert this to fluid power:

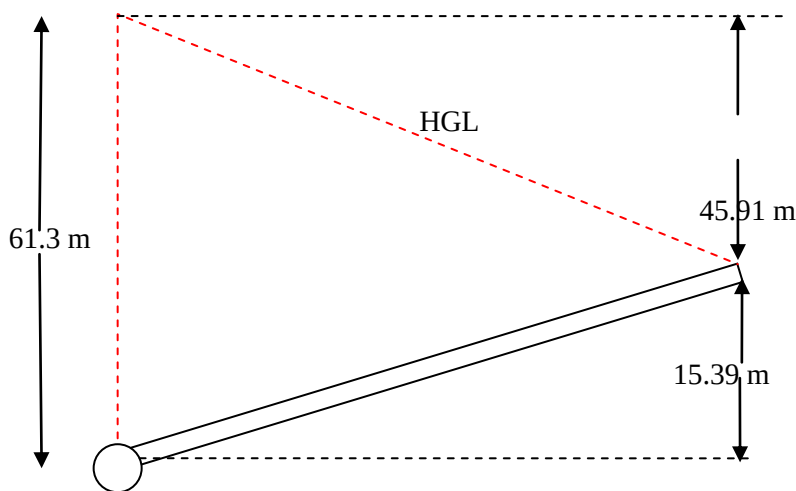
$$\begin{aligned} \text{Power} &= \rho g Q \Delta H \\ &= 1000(9.8)0.45(54) = 238.1 \times 10^3 \text{ Watts} \end{aligned}$$

A 153 mm diameter steel pipe (roughness height $\varepsilon = 0.061$ mm) is 1224 m long and rises 15.39 m. A pump of pressure rating 513 KPa is to be used to pump medium fuel oil at 21.1°C (s.g = 0.854, $\nu = 3.83 \times 10^{-6}$ m²/s). What is the maximum flow of oil the system will carry?

The pressure at the outlet from the fuel line falls to zero (atmospheric gauge pressure) at the maximum flowrate. We can work out the head loss by considering the pump capacity and the rise in the pipe:

$$\text{Pump head} = 513 \times 10^3 / \rho g = 513 \times 10^3 / (854(9.8)) = 61.3 \text{ m}$$

$$\text{Maximum Headloss} = 61.3 - 15.39 = 45.91 \text{ m}$$



$$\text{So } h_f = 45.91 = \lambda L V^2 / 2gD$$

Rearranging knowns and unknowns, we have

$$\lambda V^2 = 45.91(2gD/L) = 45.91(2(9.8)0.153/1224) = 0.1125$$

Friction factor λ and velocity V are related by the Reynold's number. We proceed by trialling values of V to calculate R and hence λ , to find a value where $\lambda V^2 = 0.1125$.

We need to allow for 3 possible ranges of Re :

$$Re < 2000: \lambda = 64/Re$$

$$2000 < Re < 10000: \lambda = 0.3164/Re^{0.25}$$

$$Re > 10000: \lambda = \left[\frac{1}{-2 \log \left[\frac{\varepsilon}{3.7D} + \frac{2.51}{Re \sqrt{\lambda}} \right]} \right]^2$$

A spreadsheet is ideal for keeping track of complex formulae and performing iterative calculations (eg where λ appears in its own equation for $Re > 10000$). In the tables below, the pipe diameter, roughness

and fluid kinematic viscosity are treated as known inputs. A low velocity is guessed and the resulting Reynolds number calculated. Friction factors for the three possible ranges are calculated (the third by trial and error), and the value corresponding to the actual Reynolds value is selected. This is then multiplied by $Velocity^2$. The process is repeated until this last calculation corresponds to the expected value of $\lambda V^2 = 0.1125$

Pipe Diameter	m	0.153
Roughness height	m	6.10E-05
Kinematic viscosity	m ² /s	3.83E-06
Trial Velocity	m/s	0.05
Reynolds number		1997
Friction factor (R<2000)		0.0320
Friction factor (2000<R<10000)		0.0473
Friction factor 3 trial		0.0498
Friction factor 3 calculated		0.0498
Friction factor		0.0320
Friction factor x Velocity ²		0.0001

no good

Trial Velocity	m/s	0.2
Reynolds number		7990
Friction factor (R<2000)		0.0080
Friction factor (2000<R<10000)		0.0335
Friction factor 3 trial		0.0334
Friction factor 3 calculated		0.0333
Friction factor		0.0335
Friction factor x Velocity ²		0.0013

still no good

Trial Velocity	m/s	1
Reynolds number		39948
Friction factor (R<2000)		0.0016
Friction factor (2000<R<10000)		0.0224
Friction factor 3 trial		0.0233
Friction factor 3 calculated		0.0232
Friction factor		0.0232
Friction factor x Velocity ²		0.0232

still not right

Trial Velocity	m/s	2
Reynolds number		79896
Friction factor (R<2000)		0.0008
Friction factor (2000<R<10000)		0.0188
Friction factor 3 trial		0.0206
Friction factor 3 calculated		0.0206
Friction factor		0.0206
Friction factor x Velocity ²		0.0823

getting there...

Trial Velocity	m/s	2.5
Reynolds number		99869
Friction factor (R<2000)		0.0006
Friction factor (2000<R<10000)		0.0178
Friction factor 3 trial		0.0199
Friction factor 3 calculated		0.0199
Friction factor		0.0199
Friction factor x Velocity ²		0.1244

nearly there...

Trial Velocity	m/s	2.368
Reynolds number		94596
Friction factor (R<2000)		0.0007
Friction factor (2000<R<10000)		0.0180
Friction factor 3 trial		0.0201
Friction factor 3 calculated		0.0201
Friction factor		0.0201
Friction factor x Velocity ²		0.1125

after about ten attempts (easy with a spreadsheet)

The velocity of 2.368 m/s gives a flowrate of

$$Q = VA = 2.368(\pi 0.153^2/4) = 0.044 \text{ m}^3/\text{s}$$

Determine the headloss over 300 m of new, uncoated 350 mm inside diameter cast iron pipe when

- Water at 20°C flows at 1.5 m/s, and
- Crude oil at 50°C flows at 0.5 m/s

We start by collecting some data on the specifications:

Water at 20°C: $\mu = 1 \times 10^{-3} \text{ Pa.s}$, $\rho = 998 \text{ kg/m}^3$

Crude oil at 50°C : $\mu = 5 \times 10^{-3} \text{ Pa.s}$, $\rho = 860 \text{ kg/m}^3$

New, uncoated cast iron pipe: $\varepsilon = 0.24 \text{ mm}$, $\varepsilon/D = 0.24/350 = 686 \times 10^{-6}$

The viscosity and density figures are used to compute Reynolds number, while the roughness height is used to calculate the friction factor at that state of flow. The roughness height ratio $\epsilon/D = 0.0007$ gives a corresponding pipe curve in the Moody diagram

The headloss is determined from the Darcy-Weisbach equation, with friction factor f as above:

$$h_f = S_f L = f \frac{L V^2}{2g D}$$

a) For water at 1.5 m/s and 20°C in a 350 mm diameter pipe, Reynolds number is:

$$Re = \frac{\rho D V}{\mu} = \frac{998 \cdot 0.350 \cdot 1.5}{1 \times 10^{-3}} = 5.24 \times 10^5$$

The Moody diagram shows an f -value of around 0.019 for $Re = 5.24 \times 10^5$. A quick check with the Colebrook-White formula confirms this:

$$\frac{1}{\sqrt{f}} = \frac{1}{\sqrt{0.01869}} = 7.315 = -2 \log \left(\frac{0.24}{3.7 \cdot 0.350} + \frac{2.51}{5.24 \times 10^5 \sqrt{0.01869}} \right)$$

So the headloss is $h_f = f \frac{L V^2}{2g D} = 0.0187(300)1.5^2 / (2(9.8)0.35) = 1.84 \text{ m}$

b) For crude oil at 0.5 m/s and 50°C, Reynolds number is:

$$Re = \frac{\rho D V}{\mu} = \frac{860 \cdot 0.350 \cdot 0.5}{5 \times 10^{-3}} = 3.01 \times 10^4$$

This corresponds to an f of around 0.025, confirmed by Colebrook-White

So the headloss is $h_f = f \frac{L V^2}{2g D} = 0.0252(300)0.5^2 / (2(9.8)0.35) = 0.275 \text{ m}$

Points 1 and 2 are 1224 m apart along a new 150 mm diameter welded steel pipe. Point 2 is 15.39 m higher than point 1. The pressures at 1 and 2 are 848 kPa and 335 kPa respectively. What flowrate can we expect for gasoline at 25°C?

Let's get some data while we calm down about the "hard" question: From the study guide the viscosity of gasoline at 25°C is 3×10^{-4} PaS and the density is 680 kg/m^3 . The roughness height of welded steel is 0.061 mm.

OK, now we write Bernoulli's equation to see how the pressure and elevation terms influence this situation:

$$\frac{V_1^2}{2g} + z_1 + \frac{P_1}{\rho g} + PumpHead - TurbineHead - HeadLoss = \frac{V_2^2}{2g} + z_2 + \frac{P_2}{\rho g}$$

There are no pumps or turbines in the system, $Z_1 = 0$ and $Z_2 = 15.39$. Also since the pipe diameter does not change, $V_1 = V_2$ and those terms cancel:

That leaves
$$z_1 + \frac{P_1}{\rho g} - HeadLoss = z_2 + \frac{P_2}{\rho g}$$

Or
$$HeadLoss = z_1 + \frac{P_1}{\rho g} - z_2 - \frac{P_2}{\rho g} = 848 \times 10^3 / (680(9.8)) - 15.39 - 335 \times 10^3 / (680(9.8)) = 61.6$$

m

This is a fairly large headloss, but then the distance is more than a kilometre so it seems reasonable.

The headloss can be equated to velocity head via the Darcy-Weisbach equation:

$$h_f = f \frac{LV^2}{2gD} = 61.6 \text{ m}$$

f and V are unknown, but we can guess a value of f , use it to calculate V and the Reynolds number, then check the corresponding f value on the Moody diagram or by the Colebrook-White formula:

Let's guess f is about 0.02:

Then
$$V^2 = h_f \frac{2gD}{fL} = 61.6(2(9.8)(0.15)) / (0.02(1224)) = 7.397 \text{ and } V = 2.72 \text{ m/s}$$

For which Reynolds number is

$$Re = \frac{\rho DV}{\mu} = \frac{680 \times 0.15 \times 2.72}{3 \times 10^{-4}} = 9.25 \times 10^5$$

For $\epsilon/D = 0.016/150 = 0.0001067$ the Moody diagram yields $f = 0.014$

Using that to repeat the calculations we find $V = 3.25 \text{ m/s}$, $Re = 1.1 \times 10^6$ and the f -value does not change substantially. So after a single iteration we find $V = 3.25 \text{ m/s}$:

$$Q = VA = 3.25\pi(0.15)^2/4 = 0.057 \text{ m}^3/\text{s}$$

After several more iterations using the Colebrook-White formula, the flowrate converges to $Q = 0.053 \text{ m}^3/\text{s}$

What size new uncoated cast iron pipe 2438 m long will deliver 1.06 m³/s of water at 20°C with a head loss of 63.53 m?

Let's get some data: $\mu = 1 \times 10^{-3}$ Pa.S, $\rho = 1000$ kg/m³, $\varepsilon = 0.24$ mm (same sources as above)

We have the headloss, but the friction factor, velocity and diameter are unknown:

$$h_f = f \frac{LV^2}{2gD} = 63.53 \text{ m}$$

Let's guess a value of D and see what comes out:

Say $D = 0.75$

Then $V = Q/A = 1.06/(\pi 0.75^2/4) = 2.399$ m/s

So $Re = VD\rho/\mu = 2.399(0.75)1000/1 \times 10^{-3} = 1.8 \times 10^6$

With $\varepsilon/D = 0.24/750 = 0.00032$ we have $f = 0.015$

So we can calculate h_f and see how close we are to an accurate guess:

$$h_f = f \frac{LV^2}{2gD} = 0.015(2438)2.399^2/(2(9.8)0.75) = 14.32 \text{ m}$$

Too small, try a smaller D to tighten up the system

Say $D = 0.6$

Then $V = Q/A = 1.06/(\pi 0.6^2/4) = 3.749$ m/s

So $Re = VD\rho/\mu = 3.749(0.6)1000/1 \times 10^{-3} = 2.2 \times 10^6$

With $\varepsilon/D = 0.24/600 = 0.0004$ we still have $f = 0.015$

$$\text{So } h_f = f \frac{LV^2}{2gD} = 0.015(2438)3.749^2/(2(9.8)0.6) = 43.71 \text{ m}$$

Try a smaller diameter again:

Try $D = 0.45$

Then $V = Q/A = 1.06/(\pi 0.45^2/4) = 6.665$ m/s

So $Re = VD\rho/\mu = 6.665(0.45)1000/1 \times 10^{-3} = 3 \times 10^6$

With $\varepsilon/D = 0.24/450 = 0.00053$ we have $f = 0.016$

$$\text{So } h_f = f \frac{LV^2}{2gD} = 0.016(2438)6.665^2/(2(9.8)0.45) = 196.5 \text{ m}$$

So between the standard sizes of 450 and 600 mm there is a non-standard pipe that will do the job.

Try $D = 0.5$

Then $V = Q/A = 1.06/(\pi 0.5^2/4) = 5.399$ m/s

So $Re = VD\rho/\mu = 5.399(0.5)1000/1 \times 10^{-3} = 2.7 \times 10^6$

With $\varepsilon/D = 0.24/500 = 0.00048$ we have $f = 0.016$

$$\text{So } h_f = f \frac{LV^2}{2gD} = 0.016(2438)5.399^2/(2(9.8)0.5) = 116.5 \text{ m}$$

Try $D = 0.55$

Then $V = Q/A = 1.06/(\pi 0.55^2/4) = 4.462 \text{ m/s}$

So $Re = VD\rho/\mu = 4.462(0.55)1000/1 \times 10^{-3} = 2.5 \times 10^6$

With $\varepsilon/D = 0.24/550 = 0.00044$ we have $f = 0.016$

So $h_f = f \frac{LV^2}{2gD} = 0.016(2438)4.462^2/(2(9.8)0.55) = 72 \text{ m}$

Try $D = 0.56$

Then $V = Q/A = 1.06/(\pi 0.56^2/4) = 4.304 \text{ m/s}$

So $Re = VD\rho/\mu = 4.304(0.56)1000/1 \times 10^{-3} = 2.4 \times 10^6$

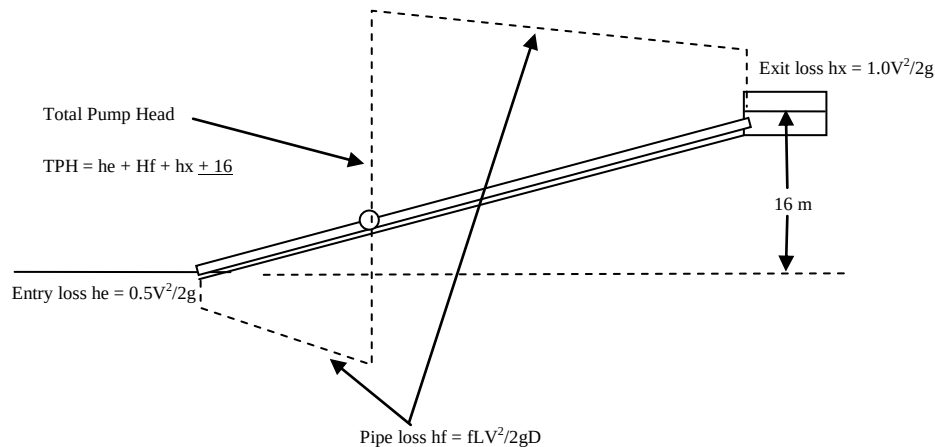
With $\varepsilon/D = 0.24/560 = 0.00043$ we have $f = 0.016$

So $h_f = f \frac{LV^2}{2gD} = 0.016(2438)4.304^2/(2(9.8)0.56) = 65.83 \text{ m}$

This is close enough for a hand calculation. With a little extra work the calculation could be set up in a spreadsheet, giving a solution to any desired accuracy. The result is a non-standard pipe size ($D = 566 \text{ mm}$) which will be expensive, so it deserves the extra work.

A pump draws water from a river at 280 m AHD (Australian Height Datum, the surveying standard height reference in Aus) and delivers it to a stock reservoir at 295 m AHD on the river bank. The reservoir has a capacity of 2 m and is usually about half full. The pump is in a pump house at 285 m AHD, connected to the river by 40 m of 30 mm diameter black PVC pipe, and to the base of the reservoir by a further 25 m of the same pipe.

- Assuming an inlet headloss factor of $0.5V^2/2g$, outlet loss of $1.0V^2/2g$ and a pipe friction factor f of 0.02, what total pump head must the pump provide to supply 4500 litres per hour to the reservoir?
- Assuming the pipe is hydraulically smooth, check that the friction factor of $f = 0.02$ is reasonable. If not, repeat the headloss calculation with the revised f .
- Repeat the calculation for a doubling of water demand in the high season. As a first guess, would it be cheaper to double the pump head or the pipe diameter to cope with this?
- What other potential problem would you have to consider with the pump located 5 m above the water surface? (no calculations, just highlight the issue).



- Convert the flowrate of 4500 litres/hour to m^3/s :

$$Q = 4500 / (1000[l/m^3] \times 3600[s/hr]) = 0.00125 \text{ m}^3/s$$

The flow area is $A = \pi D^2/4 = \pi(0.03)^2/4 = 707 \times 10^{-6} \text{ m}^2$

The flow velocity is then $V = Q/A = 1.77 \text{ m/s}$

The headloss factors can be evaluated:

Inlet loss: $h_i = 0.5V^2/2g = 0.5(1.77)^2/19.6 = 0.08 \text{ m}$

Outlet loss: $h_o = V^2/2g = 1.77^2/19.6 = 0.16 \text{ m}$

Pipe loss: Total length = 65 m, $f = 0.02$

$h_f = fLV^2/2gD$ (Darcy-Weisbach Equation)

$$= 0.02(65)1.77^2/(19.6(0.03)) = 6.9 \text{ m}$$

In many cases, as here, inlet and other losses are negligible compared to pipe friction.

The total head loss is $6.9 + 0.16 + 0.08 = 7.15 \text{ m}$

The pump must overcome this loss as well as provide a lift of $296 - 280 = 16$ m:

$$\underline{\underline{H = 16 + 7.15 = 23.15 \text{ m} \quad \text{Total Pump Head Required at 1.25 l/s flow}}}$$

b) The loss factor is a function of the Reynolds number:

$$Re = VD\rho/\mu$$

$$V = 1.77 \text{ m/s}, D = 0.03 \text{ m}, \rho = 1000 \text{ kg/m}^3, \mu = 0.001 \text{ Pa.s}$$

$$\text{Thus } Re = 1.77(0.03)1000/0.001 = 5.3 \times 10^4$$

This is in the turbulent regime, but only just. The Moody diagram for smooth pipes gives a friction factor of about $f = 0.02$. The calculation is valid.

c) Doubling the demand to 9000 l/hour, we have $Q = 0.0025 \text{ m}^3/\text{s}$ and $V = 3.54 \text{ m/s}$.

$$\underline{\text{Inlet loss:}} \quad h_i = 0.5V^2/2g = 0.5(3.54)^2/19.6 = 0.32 \text{ m}$$

$$\underline{\text{Outlet loss:}} \quad h_o = V^2/2g = 3.54^2/19.6 = 0.64 \text{ m}$$

Pipe loss: Total length = 65 m, guess $f = 0.018$ (moody diagram indicates it will drop with increasing Re)

$$h_f = fLV^2/2gD \text{ (Darcy-Weisbach Equation)}$$

$$= 0.018(65)3.54^2/(19.6(0.03)) = 24.9 \text{ m}$$

The total head loss is $24.9 + 0.32 + 0.64 = 25.9 \text{ m}$

$$\underline{\underline{H = 16 + 25.9 = 41.9 \text{ m} \quad \text{Total Pump Head Required at 2.5 l/s flow}}}$$

$$Re = 3.54(0.03)1000/0.001 = 1.1 \times 10^5 \quad f = 0.018 \text{ OK}$$

For a doubling of demand the pump head required has also nearly doubled, mainly because the pipe losses have increased by $24.9/6.9 = 3.6$ x. Doubling the pipe diameter reduces the velocity head and loss factors through the system:

$$\text{For 60 mm diameter, } A = \pi D^2/4 = \pi(0.06)^2/4 = 0.00283 \text{ m}^2$$

$$V = Q/A = 0.0025/0.00283 = 0.88 \text{ m/s}$$

$$\underline{\text{Inlet loss:}} \quad h_i = 0.5V^2/2g = 0.5(0.88)^2/19.6 = 0.02 \text{ m}$$

$$\underline{\text{Outlet loss:}} \quad h_o = V^2/2g = 0.88^2/19.6 = 0.04 \text{ m}$$

Pipe loss: Total length = 65 m, $f = 0.02$

$$h_f = fLV^2/2gD \text{ (Darcy-Weisbach Equation)}$$

$$= 0.02(65)0.88^2/(19.6(0.06)) = 0.86 \text{ m}$$

$$Re = 0.88(0.06)1000/0.001 = 5.3 \times 10^4 \quad f = 0.02 \text{ OK}$$

A doubling of pipe diameter dramatically reduces the headlosses. A smaller pump could thus be used for both regular and high flow duties. This could easily offset the cost of the pipe and seems an obvious improvement in this situation.

Careful, 60 mm pipe is less manageable on rough ground and has higher drag in flood.

In practice the optimal economic balance has to be worked out by trialling available pump head-discharge rating curves with pipe sizes and pressures that are suited to those pumps.

d) The pump is elevated 5 m above the water surface and draws the water up initially through a *suction lift*. This places the water column in tension and may reduce its pressure to near vapour pressure, depending also on losses occurring in the lift pipe. Worst case scenario is if it drops to below vapour pressure as it enters the pump: cavitation occurs, ie the water boils and collapses violently between vapour and liquid states, damaging the pump impeller.

Also the lower pipeline at the pump will drain between uses, so the pump line will need to be *primed* (filled with water to create a water column to the river level) before use.

Also the pump and pump house may be damaged during flood.

Drag, Lift and Aerodynamics

A long copper wire, 12 mm in diameter, is stretched taut and exposed to 10°C wind at a velocity of 27.5 m/s. What is the drag force per metre length of wire?

We can consider the wire as an “infinite cylinder”, to use the language of the drag coefficient diagram.

We need to work out the Reynolds number of the wire in this flow:

$$Re = \frac{\rho DV}{\mu}$$

The density of air in normal conditions is close enough to 1.2 kg/m³. For a more accurate estimate, use $P = \rho RT$: $\rho = P/RT = 101300/(287(273+10)) = 1.247 \text{ kg/m}^3$. Let's go with that value.

The viscosity of air at 10°C is $1.8 \times 10^{-5} \text{ Pa.s}$ (p20 Study Guide)

So with a diameter of 0.012 m and velocity of 27.5 m/s we have

$$Re = \frac{\rho DV}{\mu} = 1.247(0.012)27.5/1.8 \times 10^{-5} = 22860$$

The drag coefficient diagram gives a C_D of 1.3 for an “infinite cylinder” at this Re .

The profile area of the wire per metre is $A = D(1) = 0.012 \text{ m}^2$

The drag force per metre can then be calculated as

$$Drag = C_D \rho \frac{U^2}{2} A = 1.3(1.247)27.5^2(0.012)/2 = \underline{7.356 \text{ N}}$$

A subject weighing 76 kg tests a 8.4 m diameter parachute. At what terminal velocity will the subject hit the ground? Assume $\rho = 1.2 \text{ kg/m}^3$ and $\mu = 2 \times 10^{-5} \text{ Pa.s}$.

We need a drag force equal to 76 kg of mass based on a fixed velocity. The actual force is

$$D = mg = 76(9.8) = 745 \text{ N}$$

The area of the chute is $\pi D^2/4 = \pi(8.4^2)/4 = 55.42 \text{ m}^2$

Then the drag force is:

$$Drag = C_D \rho \frac{U^2}{2} A = C_D(1.2)V^2(55.42/2) = 33.25C_D V^2 = 745 \text{ N}$$

For circular disks $C_D = 1.12$ (study guide)

Then $V = (745/(33.25(1.12)))^{0.5} = 4.473 \text{ m/s}$ (similar speed to jumping off a fence)

A steel ball 3 mm in diameter falls through an oil bath of density 908 kg/m^3 and kinematic viscosity $1.46 \times 10^{-4} \text{ m}^2/\text{s}$. What is the terminal velocity of the ball? Density (steel) = 7700 kg/m^3

The ball's mass is $m = \rho V = \rho(4/3 \pi D^3/8) = 7700(4/3\pi(0.003^3)/8) = 108.9 \times 10^{-6} \text{ kg}$ (about 0.1 g)

The ball's profile area is $A = \pi D^2/4 = \pi(0.003^2)/4 = 7.069 \times 10^{-6} \text{ m}^2$

The terminal velocity occurs when the drag force $D =$ the weight force $W = mg = 1.067 \times 10^{-3} \text{ N}$

$$Drag = C_D \rho \frac{U^2}{2} A = C_D(908)V^2(7.069 \times 10^{-6}/2) = 3.209 \times 10^{-3} C_D V^2 = 1.067 \times 10^{-3} \text{ N}$$

So we need $C_D V^2 = 1.067 \times 10^{-3} / 3.209 \times 10^{-3} = 0.332$

We can proceed iteratively by guessing a value of C_D , calculating V and Re to locate a value of C_D , and repeating to convergence:

Say $C_D = 1$

Then $V = (0.332)^{0.5} = 0.576 \text{ m/s}$

$Re = 0.576(0.003)/1.46 \times 10^{-4} = 11.84$ (laminar)

For spheres in this range we have $C_D = 4.5$

Then $V = (0.332/4.5)^{0.5} = 0.272 \text{ m/s}$

$$Re = 0.272(0.003)/1.46 \times 10^{-4} = 5.59 \text{ (laminar)}$$

For spheres in this range we have $C_D = 7$

$$\text{Then } V = (0.332/7)^{0.5} = 0.218 \text{ m/s}$$

$$Re = 0.218(0.003)/1.46 \times 10^{-4} = 4.09 \text{ (laminar)}$$

For spheres in this range we have $C_D = 7.5$

$$\text{Then } V = (0.332/7.5)^{0.5} = 0.210 \text{ m/s}$$

The solution converges on a velocity of 0.2 m/s

An airplane weighs 1.837 tonnes and has a wing area of 28 m^2 . What angle of attack must the wings make with the horizontal for a 160 km/h cruising speed? Use $\rho = 1.2 \text{ kg/m}^3$ and the “typical” lift and drag coefficients in the lectures.

Let's convert that airspeed to SI units first of all: $V = 160/3.6 = 44.44 \text{ m/s}$

Now let's look at the aerodynamic lift equation:

$$Lift = C_L \rho \frac{U^2}{2} A_w$$

The lift is an equilibrium force, equal to the aircraft's weight in Newtons. Convert the mass to weight:

$$Lift = weight = mg = 1837(9.8) = 18 \times 10^3 \text{ N}$$

All that remains is the lift coefficient C_L :

$$C_L = \frac{Lift}{\rho \frac{U^2}{2} A_w} = 18 \times 10^3 / (1.2(44.44^2)28/2) = 0.5425$$

The diagram gives a lift coefficient of approximately 0.55 for an angle of attack of 5.0° . This is a suitable angle of attack within the accuracy of this model.

What wing area is required to support a 2.2 tonne aircraft cruising at 27m/s when the wing has a 5° angle of attack?

We can go straight to the lift equation:

$$Lift = C_L \rho \frac{U^2}{2} A_w$$

Where C_L is a lift coefficient (assume the diagram on p 149 Study Guide is accurate).

At cruising speed the lift must equal the weight: $Lift = mg = 2.2 \times 10^3(9.8) = 21.56 \times 10^3 \text{ N}$

At 5° the lift coefficient is 0.65 (Study Guide)

$$\text{So } C_L \rho U^2 A_w / 2 = 0.65(1.2)27^2 A_w / 2 = 21.56 \times 10^3 \text{ N}$$

$$\text{Thus } A_w = 21.56 \times 10^3 (2) / (0.65(1.2)27^2) = 75.83 \text{ m}^2$$

A kite weighs 1.12 kg and has an area of 0.75 m^2 . On a day when the wind speed is 9 m/s the kite rises at an angle of attack of 8° to the horizontal, reaching a maximum height when the string makes an angle of 45° to the horizontal, at which point the tension in the kite is measured as 30 N. What are the approximate drag and lift coefficients for this simple wing?

At the cruising altitude the lift force = the weight force + the vertical component of the tension, and the drag force = the horizontal component of the tension:

$$\text{Lift} = 1.12(9.8) + 30\sin(45) = 32.19 \text{ N}$$

$$\text{Drag} = 30\cos(45) = 21.21 \text{ N}$$

We need to resolve the kite area into horizontal (drag) and vertical (lift) components*:

$$A_L = 0.75\cos(8) = 0.742 \text{ m}^2$$

$$A_D = 0.75\sin(8) = 0.104 \text{ m}^2$$

Then the lift and drag equations are

$$\text{Lift} = C_L \rho \frac{U^2}{2} A_L = C_L(1.2)9^2 0.742 / 2 = 32.19 \text{ N}$$

$$\text{So } C_L = 32.19(2) / (0.742(9^2)1.2) = 0.893$$

$$\text{And } \text{Drag} = C_D \rho \frac{U^2}{2} A_D = C_D(1.2)9^2 0.104 / 2 = 21.21 \text{ N}$$

$$\text{So } C_D = 21.21(2) / (0.104(9^2)1.2) = 4.196$$

*This is a more accurate way of representing the area of a wing in its two principle profile areas. Published values of C_L and C_D may or may not use this formulation.

The basic specifications for an F14 jet are as follows:

- Wingspan 64 m spread, 38 m swept
- Total Wing area $A_w = 54.5 \text{ m}^2$
- Drag profile area $A_p = 11.5 \text{ m}^2$
- Mass 27,700 kg fully loaded
- 2 x GE F110 Turbofan engines
- Thrust 61.4 kN from each engine, 124.7 kN each with afterburners lit

- Maximum airspeed Ma2.34 or 2485 km/h at high altitude
- Peak Thrust/Weight ratio 0.91
- a) Using the data, derive a drag coefficient for the F14 at maximum thrust (with afterburners) and airspeed. Assume air temperature -56.5°C and pressure 26 kPa absolute at 10 km alt.
- b) The lift coefficient at top airspeed is rated at 0.054. What lift does that generate? Is the fully loaded aircraft climbing or descending at 10 km altitude and top speed?
- c) A 1:32 scale model F14 weighs 2.5 kg. What are the ratios of wing span, area and weight to the full scale? Assuming the drag and lift coefficients are in similar ratios to the peak performance coefficients in a) and b), what thrust does the model require at cruising speed?
- d) Given c) and assuming a model drag coefficient 100x that of the prototype peak performance drag coefficient, what is the model's cruising speed near sea level at 15°C?

A diagram adds nothing to this, but you should make a quick sketch so you don't mix A_w with A_p etc.

a) Thrust = $2 \times 124.7 = 249.4 \text{ kN} = \text{Drag at top speed (Thrust = Drag at any constant V)}$

$$\text{Drag} = C_d \rho V^2 A_p / 2 = 249.4 \times 10^3 \text{ N}$$

$$V = 2485 \text{ [km/h]} \times 1000 \text{ [m/km]} / 3600 \text{ [s/h]} = 690.3 \text{ m/s}$$

$$A_p = 11.5 \text{ m}^2 \text{ (given)}$$

$$\rho = P/RT = 26 \times 10^3 / (287(273.3 - 56.5)) = 0.42 \text{ kg/m}^3$$

Thus $C_d = 249.4 \times 10^3 / (0.42(690.3^2)11.5/2) = 0.217$ (dimensionless)

b) Lift = $C_L \rho V^2 A_w / 2 = 0.054(0.42)690.3^2(54.5)/2 = 294500 = 294.5 \text{ kN}$

The loaded aircraft weighs 27700 kg or $27700(9.8) = 271.5 \text{ kN}$

The lift exceeds the load, so the aircraft is still climbing at this speed and altitude.

c) The wing span ratio is just the linear scale, $L_r = 1/32 = 0.03125$

The wing area is much less than this: $A_r = 1/32^2 = 0.000976 = L_r/32$

The weight ratio is smaller still: $W_r = 2.5/27700 = 0.00009025 = L_r/346.3$

At cruising speed, Lift = Weight and Drag = Thrust

We don't know the lift or drag coefficients, but we are guessing (as advised) that they are in a similar ratio to before:

$$C_D/C_L = 0.217/0.054 = 4.02$$

We can form similar ratios of weight/thrust:

$$\text{Thrust} = C_d \rho V^2 A_p / 2 = \text{Drag}$$

$$\text{Weight} = C_L \rho V^2 A_w / 2 = \text{Lift}$$

$$\text{So Thrust/Weight} = C_d A_p / C_L A_w = C_d / C_L \times A_p / A_w = 4.02 \times 11.5 / 54.5 = 0.85$$

$$\text{And Thrust} = 0.85 \times \text{Weight} = 0.85 \times 9.8 \times 2.5 = \underline{\underline{20.8 \text{ N}}}$$

This is about 2.1 kg of thrust, or a thrust to weight ratio of 0.84. The data show a peak thrust/weight ratio of 0.91 so this seems reasonable, if a bit high for minimum cruising speed. Drag and thrust coefficients don't necessarily have similar ratios at different Reynolds numbers.

Note that model and prototype aircraft DO NOT fly at the conditions suggested by similitude, these are laboratory settings only. Actual flight settings for a model are worked out separately from the prototype.

d) Assuming $C_d = 100(0.217) = 21.7$ (reasonable given the rapid increase in C_d for small Re)

$$\text{Drag} = C_d \rho V^2 A_p / 2 = \text{Thrust} = 20.8 \text{ N}$$

$$A_p = 11.5/32^2 = 0.0112 \text{ m}^2$$

$$\rho = P/RT = 101325/(287(288)) = 1.23 \text{ kg/m}^3 \text{ (normal air density)}$$

$$\text{So } V = (2(\text{Thrust})/(C_d \rho A_p))^{1/2} = (2(20.8)/(21.7(1.23)0.0112))^{1/2} = 11.8 \text{ m/s} = \underline{\underline{42.5 \text{ km/h}}}$$

Velocities much larger or smaller than this would not be applicable to a model at normal cruising flight speeds but to laboratory simulations with a fixed support for the model. Check your answer to decide if say 2 m/s or 200 m/s are reasonable cruise speeds for a 2-m wingspan model plan.