
Tutorial 4 Momentum Conservation and Turbomachinery

Momentum Conservation

A jet of water 75 mm in diameter hits a flat plate held at right angles to the jet. If the jet's velocity is 25 m/s, what force is required to hold the plate in a fixed position?

We can simplify things a bit by ignoring the issue of deflection, which will have the effect of spraying water at high velocities at 90° in all directions, and just think about the change in velocity we are trying to create in the direction of the plate: that is from 25 m/s to zero. This is a fairly large impact and will generate a correspondingly large force.

We have the momentum reaction law in its simplest form for assumed uniform flow conditions:

$$\sum \underline{F} = \dot{m} \Delta \underline{V}$$

Where $\dot{m}$ is the mass flow rate

$$\dot{m} = \rho Q$$

The flowrate Q can be determined from the jet diameter and velocity:

$$Q = VA = 25\pi(0.075^2/4) = 0.110 \text{ m}^3/\text{s}$$

So $\dot{m} = \rho Q = 1000(0.110) = 110 \text{ kg/s}$

And since $\Delta V = 25 \text{ m/s}$ we have

$$F = 110(25) = 2750 \text{ N}$$

A 600 mm pipe carrying $0.9 \text{ m}^3/\text{s}$ of oil, density 850 kg/m^3 , has a 90° horizontal bend. Determine the reaction force magnitude and direction.

In this case the ΔV term is more obviously a vector quantity, as the speed doesn't change but the direction does. Let's work out the speed and then turn it into a vector (see the Vector Calculus Refresher Tutorial if necessary):

$$V = Q/A = 0.9/(\pi(0.6^2/4)) = 3.183 \text{ m/s}$$

Let's say this is in the x-direction, with no component in the y-direction.

So $\underline{V}_1 = (3.183, 0)$

After the 90° bend the speed is unchanged, but has moved wholly from the x to the y-direction:

Ie $\underline{V}_2 = (0, 3.183)$

Now in vector terms $\Delta \underline{V} = (\underline{V}_2 - \underline{V}_1, \underline{V}_2 - \underline{V}_1) = (0 - 3.183, 3.183 - 0) = (-3.183, 3.183)$

We can multiply any vector by a scalar number, say the mass flow rate:

$$\dot{m} = \rho Q = 850(0.9) = 765 \text{ kg/s}$$

Then the force, as a vector, is just

$$\underline{\mathbf{F}} = \dot{m} \Delta \underline{\mathbf{V}} = 765(-3.183, 3.183) = (-2448, 2448)$$

This is the force of the bend on the fluid, which decelerates as the x-velocity goes from 3.2 to 0, and accelerates as the y-velocity goes from 0 to 3.2 m/s. The force has equal components in the x and y-directions. The reaction force, of the fluid acting on the bend, is equal and opposite:

$$\underline{\mathbf{F}}_r = -\underline{\mathbf{F}} = (2448, -2448)$$

The magnitude of the reaction force is $|\underline{\mathbf{F}}| = \sqrt{(F_x)^2 + (F_y)^2} = (2(2448^2))^{0.5} = 3.46 \times 10^3 \text{ N}$

And the direction is $\theta = \tan^{-1}(F_y/F_x) = \tan^{-1}(1) = 45^\circ$

A 45° reducing bend, 600 mm upstream and 300 mm downstream, has 0.45 m³/s of water flowing through it. The gauge pressure upstream is 145x10³ Pa. Determine the total force including pressure forces on the bend.

In many situations we have to take into account the loads that pressure exerts on bends of varying geometry. In a static situation for instance, pressure through a bend would tend to try to straighten the bend, much like a party whistle that unrolls when you blow into it. In a dynamic situation, the upstream pressure will change as the flow accelerates according to Bernoulli's principle, so we are going to have to deal with changing pressures and velocities, and work out the forces they create in different directions.

Let's work out the velocities first:

$$V = Q/A$$

$$A_1 = \pi 0.6^2/4 = 0.283 \text{ m}^2$$

$$A_2 = \pi 0.3^2/4 = 0.071 \text{ m}^2$$

$$\text{So } V_1 = 0.45/0.283 = 1.59 \text{ m/s}$$

$$\text{And } V_2 = 0.45/0.071 = 6.338 \text{ m/s}$$

Bernoulli's equation with no elevation changes or losses:

$$\frac{V_1^2}{2g} + \frac{P_1}{g\rho_1} = \frac{V_2^2}{2g} + \frac{P_2}{g\rho_2}$$

$$P_2 = \rho g \left(\frac{V_1^2}{2g} + \frac{P_1}{g\rho_1} - \frac{V_2^2}{2g} \right) = 9800(1.59^2/19.62 + 145 \times 10^3/9800 - 6.338^2/19.62) = 126.2 \times 10^3 \text{ Pa}$$

These pressures exert force against their respective areas:

$$F_1 = P_1 A_1 = 145 \times 10^3 (0.283) = 41.04 \times 10^3 \text{ N}$$

$$F_2 = P_2 A_2 = 126.2 \times 10^3 (0.071) = 8.96 \times 10^3 \text{ N}$$

F_1 can be regarded as a vector directed in the x-direction. As a vector, $\underline{F}_1 = (41.04 \times 10^3, 0)$

F_2 is directed back against the pipe axis at -45° to the x and y directions. This can be resolved into components as $|\underline{F}| = \sqrt{(F_x^2 + F_y^2)} = \sqrt{(2F_x^2)}$

$$\text{Thus } F_x = F_y = \sqrt{(F^2/2)} = ((8.96 \times 10^3)^2/2)^{0.5} = 6.336 \times 10^3 \text{ N}$$

$$\text{So } \underline{F}_2 = (-6.336 \times 10^3, -6.336 \times 10^3)$$

The combined pressure force is the sum of $\underline{F}_1$ and $\underline{F}_2$:

$$\underline{F}_p = \underline{F}_1 + \underline{F}_2 = (41.04 \times 10^3 - 6.336 \times 10^3, 0 - 6.336 \times 10^3) = (34.7 \times 10^3, -6.336 \times 10^3) \text{ N}$$

The momentum reaction law is the second part of this problem: still working in vectors,

$$\underline{V}_1 = (1.59, 0)$$

After the 45° bend the speed is increased, and shifted halfway to the y-direction:

$$V_x = V_y = \sqrt{(V^2/2)} = (6.338^2/2)^{0.5} = 4.482 \text{ m/s}$$

$$\text{Ie } \underline{V}_2 = (4.482, 4.482)$$

$$\text{In vector terms } \Delta \underline{V} = (V_{x2} - V_{x1}, V_{y2} - V_{y1}) = (4.482 - 1.59, 4.482) = (2.892, 4.482) \text{ m/s}$$

Then the mass flow rate is:

$$\dot{m} = \rho Q = 1000(0.45) = 450 \text{ kg/s}$$

Then the fluid force, as a vector, is just

$$\underline{F} = \dot{m} \Delta \underline{V} = 450(2.892, 4.482) = (1301, 2017) \text{ N}$$

This is the force of the bend on the fluid. The reaction force, of the fluid acting on the bend, is equal and opposite:

$$\underline{F}_r = -\underline{F} = (-1301, -2017)$$

The total force of pressure and reaction is the sum of the two vector forces:

$$\underline{F}_t = \underline{F}_r + \underline{F}_p = (34.7 \times 10^3, -6.336 \times 10^3) + (-1301, -2017) = (33.4 \times 10^3, -8.353 \times 10^3) \text{ N}$$

This has magnitude and direction:

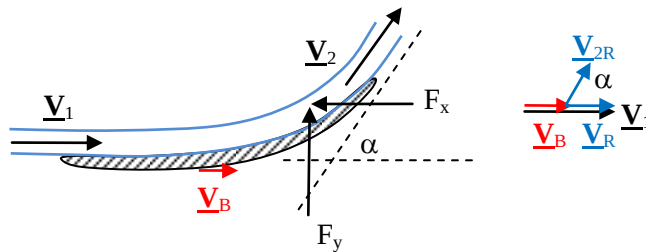
$$F_t = |\underline{F}| = \sqrt{(F_x^2 + F_y^2)} = ((33.4 \times 10^3)^2 + (-8.353 \times 10^3)^2)^{0.5} = 34.43 \times 10^3 \text{ N}$$

$$\theta = \tan^{-1}(F_y/F_x) = \tan^{-1}(-8.353 \times 10^3 / 33.4 \times 10^3) = -14.04^\circ$$

This is to the right and downwards. As expected, the bend will be forced in a direction that tends to straighten it out.

A 75 mm diameter jet has a velocity of 35 m/s. It strikes a blade moving in the same direction at 20 m/s. The deflection angle of the blade is 150° . Assuming no friction, calculate the X and Y components of the force exerted by the water on the blade. (Giles 11.8) Assignment 3 2010

From the study guide for moving vanes and deflectors, we have:



The deflection angle of 150° is not shown to scale. Note that for angles $> 90^\circ$ the cosine of the angle is negative and must be used as such in the formula.

$$F_x = \dot{m}_R \Delta V_{xR} = \dot{m}_R V_R (\cos \alpha - 1)$$

$$F_y = \dot{m}_R \Delta V_{yR} = \dot{m}_R V_R \sin \alpha$$

With a relative velocity of $\underline{V}_R = \underline{V}_1 - \underline{V}_B$ and a relative mass flowrate of $\dot{m}_R = \dot{m} \left(\frac{V_R}{V_1} \right)$ we have

$$V_R = 35 - 20 = 15 \text{ m/s}$$

$$\dot{m} = \rho V_1 A = \rho V_1 \pi D^2 / 4 = 1000(35)\pi(0.075)^2 / 4 = 154.6 \text{ kg/s}$$

$$\dot{m}_R = 154.6(15)/35 = 66.27 \text{ kg/s}$$

$$\text{So } F_x = \dot{m}_R \Delta V_{xR} = \dot{m}_R V_R (\cos \alpha - 1) = 66.27(15)(\cos 150 - 1) = -1855 \text{ N}$$

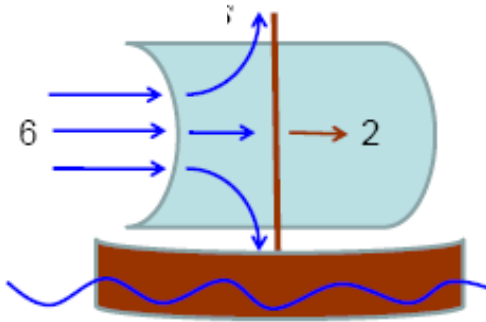
$$\text{And } F_y = \dot{m}_R \Delta V_{yR} = \dot{m}_R V_R \sin \alpha = 66.27(15)\sin 150 = 497.0 \text{ N}$$

These are the forces exerted by the vane on the water: F_x is -ve because the water is decelerated horizontally (even when the deflection angle is $< 90^\circ$) and F_y is positive because the water is accelerated vertically upwards. The corresponding forces of the water on the vane are the negatives of these values as the vane is forced to the right and downwards:

$$F_x = 1855 \text{ N}; F_y = -497 \text{ N} \quad \text{Forces exerted by water on vane.}$$

A square-rigged sail 4m x 8m in size powers a Viking longboat downwind at 2 m/s in a 6 m/s breeze. Neglecting edge effects, calculate the force exerted by the wind on the sail. (new original) Assignment 3 2010

Neglecting edge effects means in effect “don’t worry about the details of how the wind actually flows around the sail” and permits us to make simplifying assumptions about the nature of the jet flow and its deflection. A simple diagram as usual is helpful:



The Vikings on board would feel a breeze at the relative velocity of $V_R = 6 - 2 = 4$ m/s, with corresponding forces on all free surfaces including the sail. The sail area is $A = 4 \times 8 = 32$ m². Neglecting edge effects we can picture the sail as a square vane intercepting a jet flow of that size, and deflecting it by 90°. This gives us a relative mass flowrate and deflection angle as per the study guide:

$$\dot{m} = \rho V_1 A = 1.2(6)32 = 230.4 \text{ kg/s}$$

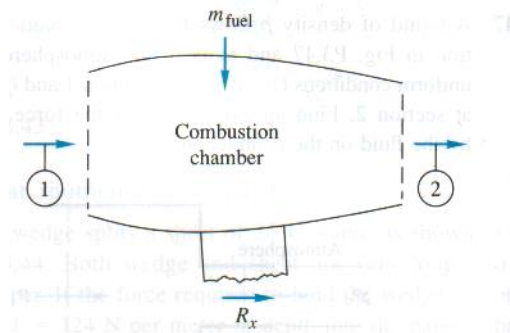
$$\dot{m}_R = \dot{m} \left(\frac{V_R}{V_1} \right) = 230.4(4)/6 = 153.6 \text{ kg/s}$$

$$\begin{aligned} F_x &= \dot{m}_R \Delta V_{xR} = \dot{m}_R V_R (\cos \alpha - 1) \\ &= 153.6(4)(\cos 90 - 1) = -614.4 \text{ N} \end{aligned}$$

This is the force exerted by the sail on the wind. The corresponding force on the sail is the negative of this value, forcing the ship to the right:

$$F = 614.4 \text{ N} \quad \text{Force of wind on sail.}$$

A jet engine on a test stand (Figure 5) intakes air at 20°C and 1 atmosphere pressure, say 100,000 Pa. The inlet aperture is 0.5 m². The intake and outlet velocities are measured at $V_1 = 250$ m/s and $V_2 = 900$ m/s. Calculate the horizontal reaction force R_x on the test stand. The engine operates at a fuel to air ratio of 1/30 (hint: add that ratio into the mass flowrate).



$$T_a = 20^\circ\text{C} = 293 \text{ K}$$

$$P_a = 100,000 \text{ Pa}$$

$$\rho_a = P/RT = 100000/(287 \times 293) = 1.19 \text{ kg/m}^3$$

$F = m \cdot \Delta V$ where m is the mass flowrate

$m = \rho Q$ where Q is the volume flowrate.

$$Q = VA = 250 (0.5) = 125 \text{ m}^3/\text{s}$$

$$m = 1.19(125) = 149 \text{ kg/s}$$

$$\text{Add in the fuel-air ratio: } m = 149(1 + 1/30) = 154 \text{ kg/s}$$

$$\Delta V = 900 - 250 = 650 \text{ m/s}$$

$$\text{Thus } F = 154 (650) = 99910 = 100 \text{ kN}$$

Turbomachinery

A turbine running at 100 RPM produces 107 kW of power under a head of 7.6 m. What power would be produced under a head of 11 m assuming the same flow? At what speed should the turbine run at the higher head to achieve this power?

The power developed is given by

$$\text{Power} = \rho g Q \Delta H = 107 \times 10^3 \text{ W}$$

The key unknown is the flowrate Q :

$$Q = P/\rho g H = 107 \times 10^3 / (1000(9.8)7.6) = 1.437 \text{ m}^3/\text{s}$$

With the same flowrate and a head of 11 m, the power would be:

$$P = 1000(9.8)1.437(11) = 154.9 \times 10^3 \text{ W} = 154.9 \text{ kW}$$

This output assumes a turbine speed which is homologous to the test setting, for which we require the turbine's specific speed:

$$N_s = \frac{N \sqrt{Power}}{H^{5/4}} = \frac{100 \sqrt{107 \times 10^3}}{7.6^{5/4}} = 2592 \text{ (study guide)}$$

The same specific speed calculation applies to the higher head with an adjusted speed:

$$N_s = 2592 = \frac{N \sqrt{154.9 \times 10^3}}{11^{5/4}}$$

$$\text{Thus } N = 2592(11^{5/4})/(154.9 \times 10^3)^{0.5} = 132 \text{ RPM}$$

Develop a theoretical expression for the head developed by a radial pump given a specification of impeller radii and operating speed.

From the Study Guide we have

$$T = \dot{m} (r_1 V_1 \cos \alpha_1 - r_2 V_2 \cos \alpha_2)$$

Where T is the torque of the impeller on the fluid and the subscripts 1 and 2 refer to major and minor impeller radii (for a pump: 1 = outflow, 2 = inflow; the reverse for a turbine).

The power of the rotation is given by the Torque multiplied by the angular velocity:

$$\text{Power} = T\omega = \dot{m} \omega (r_1 V_1 \cos \alpha_1 - r_2 V_2 \cos \alpha_2)$$

Power is also a function of head gain or loss:

$$\text{Power} = \rho g Q H = \dot{m} g H$$

The head gain through a pump is thus

$$H = \text{Power} / (\dot{m} g) = \omega (r_1 V_1 \cos \alpha_1 - r_2 V_2 \cos \alpha_2) / g$$

Note the ω is the circumferential velocity of the impeller, often denoted u:

$$H = (u_1 V_1 \cos \alpha_1 - u_2 V_2 \cos \alpha_2) / g$$

This is recognisable as a form of velocity head, v^2/g

A 3 m impeller turbine develops 5400 kW at 200 RPM under a head of 240 m and an efficiency of 82%. a) Calculate the head and discharge relationships and specific speed for the turbine. b) For this turbine, what would be the speed, flowrate and power for a head of 160 m? c) For a homologous unit having the same design, what diameter, speed and flowrate would be required to generate 2850 kW at 183 m head?

a) The head and discharge relationships and specific speed give dimensionless numbers that are common to all homologous units of this design, operating at comparable flowrates and heads.

$$\frac{Q}{ND^3} = \text{constant } t \text{ (discharge relationship) and } \frac{H}{N^2 D^2} = \text{constant } t \text{ (head relationship)}$$

$$N_s = \frac{N\sqrt{Q}}{H^{3/4}} = \frac{N\sqrt{\text{Power}/H}}{H^{3/4}} = \frac{N\sqrt{\text{Power}}}{H^{5/4}} \text{ Turbine specific speed}$$

We are not told the flowrate for the design parameters, but this can be determined from the head and power as

$$\text{Power} = \zeta \rho g Q H$$

Where ζ is the efficiency.

$$\text{So } Q = \text{Power}/\zeta \rho g H = 5400 \times 10^3 / (9800(240)0.82) = 2.8 \text{ m}^3/\text{s}$$

$$\text{Thus } \frac{Q}{ND^3} = 2.8/(200(3^3)) = 518.5 \times 10^{-6} \text{ (no units)}$$

$$\frac{H}{N^2 D^2} = 240/(200^2 3^2) = 666.7 \times 10^{-6} \text{ (no units)}$$

$$\text{And } N_s = \frac{N\sqrt{\text{Power}}}{H^{5/4}} = 200(5400 \times 10^3)^{0.5}/240^{5/4} = 492 \text{ RPM}$$

b) For a head of 160 m we have the same Head relationship at a homologous speed:

$$\frac{H}{N^2 D^2} = 160/(N^2 3^2) = 666.7 \times 10^{-6}$$

$$\text{Thus } N = (160/(666.7 \times 10^{-6} 3^2)) = 163 \text{ RPM}$$

This speed gives the homologous flowrate:

$$\frac{Q}{ND^3} = Q/(163(3^3)) = 518.5 \times 10^{-6}$$

$$\text{Thus } Q = 163(3^3) 518.5 \times 10^{-6} = 2.28 \text{ m}^3/\text{s}$$

The specific speed gives us the power at the new speed:

$$N_s = \frac{N\sqrt{Power}}{H^{5/4}} = 163(Power)^{0.5}/160^{5/4} = 492 \text{ RPM}$$

Thus $Power = (492(160^{5/4})/163)^2 = 2.950 \times 10^6 \text{ W} = 2950 \text{ kW}$

c) For a power of 2850 kW at 183 m head we should find the same specific speed:

$$N_s = \frac{N\sqrt{Power}}{H^{5/4}} = N(2850)^{0.5}/183^{5/4} = 492 \text{ RPM}$$

Thus $N = 492(183^{5/4})/(2850)^{0.5} = 196 \text{ RPM}$

The head relationship leads us to the target diameter:

$$\frac{H}{N^2 D^2} = 183/(196^2 D^2) = 666.7 \times 10^{-6}$$

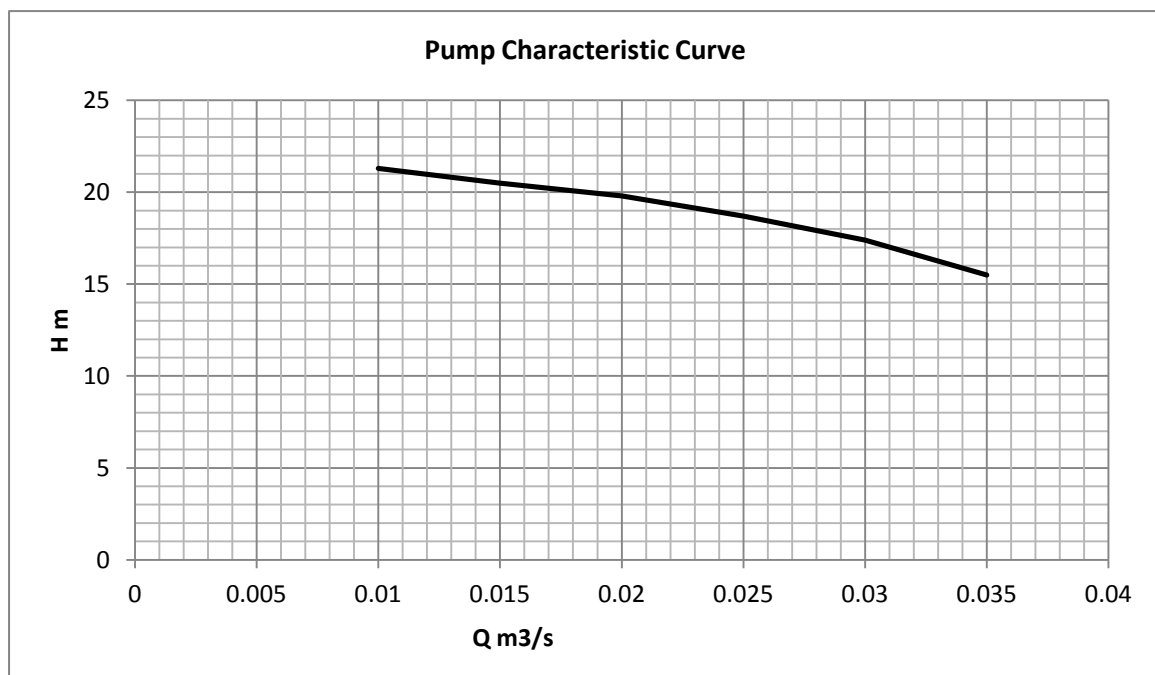
Thus $D = (183/(666.7 \times 10^{-6}(196^2)))^{0.5} = 2.673 \text{ m}$

And the discharge relationship gives the target discharge:

$$\frac{Q}{ND^3} = Q/(200(2.673^3)) = 518.5 \times 10^{-6}$$

Thus $Q = 518.5 \times 10^{-6}(200)2.673^3 = 1.98 \text{ m}^3/\text{s}$

A pump running at 1750 RPM has a head discharge curve as shown below. The pump is designed to pump water through 450 m of 150 mm pipe, at a friction factor $f = 0.025$. The pipe rises 10m over that distance. Calculate the flowrate and pump head for this system. (Giles 12.21) Tutorial 4 2010



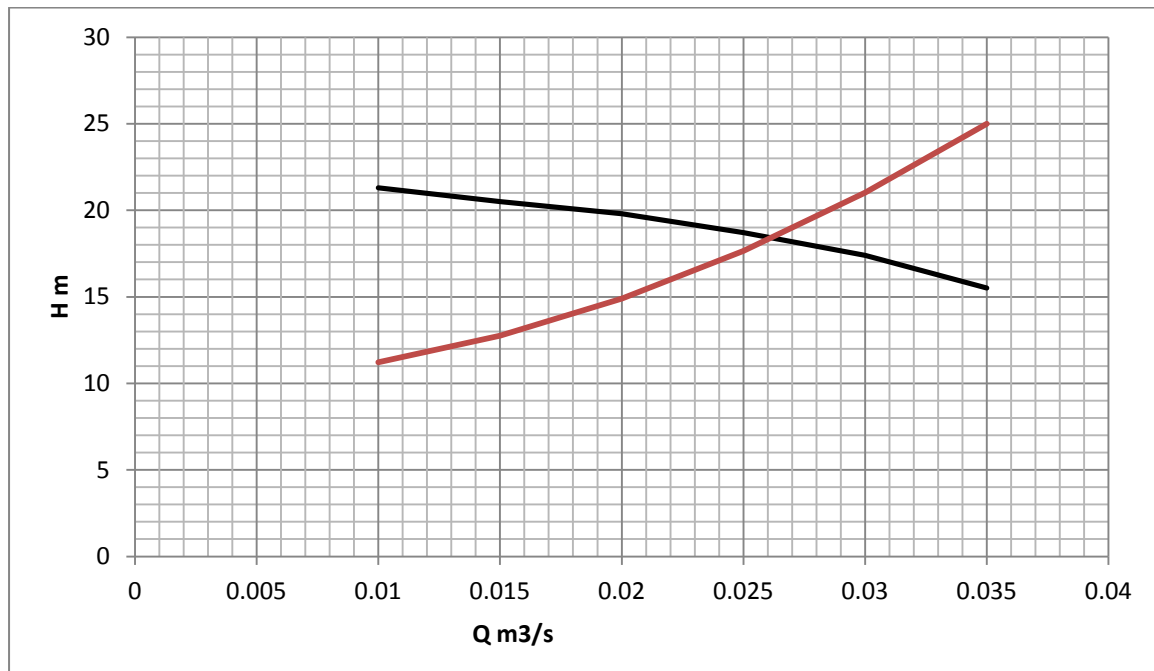
The pump must provide sufficient head to overcome the rise in pipe elevation and the headloss. We can take a shortcut past Bernoulli's equation by considering only those terms:

$$H_p = 10 + h_f = 10 + fLV^2/2gD = 10 + 0.025(450)V^2/(19.62(0.15)) = 10 + 3.823V^2$$

We can tabulate V and H_p as a function of Q (a system curve) and locate the intersection with the pump characteristic curve:

Q	m ³ /s	0.01	0.015	0.02	0.025	0.03	0.035
V = Q/A	m/s	0.566	0.849	1.132	1.415	1.698	1.981
3.823V ²	m	1.22	2.76	4.9	7.65	11.02	15.00
H _p	m	11.22	12.76	14.9	17.65	21.02	25.00

These figures can be superimposed on the characteristic curve and the intersection found directly:



The solution is at a flowrate of 0.026 m³/s where the pump head is 18 m.

A 300 mm model pump is tested at its recommended operating speed of 1500 RPM and head of 4 m and is found to deliver a flowrate of 40 litres/s. Using the Discharge and Head Relationships based on model performance data, estimate the diameter and speed of a homologous larger unit to efficiently pump 180 litres per second at a head of 7 m. Check that the Head and Discharge Relationships of the resulting design are the same as the model. Can you understand why the diameter increases but the speed decreases in this situation? (Ass3 2012)

Make sure you convert units to the same distance measure – metres, m³

Head relationship: $H_r = H/N^2 D^2 = 4/(1500^2 (0.3)^2) = 19.8 \times 10^{-6}$

Discharge Relationship: $Q_r = Q/ND^3 = 0.04/(1500(0.3)^3) = 987 \times 10^{-6}$

Eliminate N: $Q_r^2 = Q^2/N^2 D^6$

Thus $N^2 = Q^2/Q_r^2 D^6 = H/H_r D^2$

So $D^6/D^2 = D^4 = Q^2/Q_r^2 \times H_r/H$

or $D = (Q^2/Q_r^2 \times H_r/H)^{1/4}$

And $N = Q/Q_r D^3$

The required diameter **$D = (Q^2/Q_r^2 \times H_r/H)^{1/4} = ((0.18/987 \times 10^{-6})^2 \times 19.8 \times 10^{-6}/7)^{1/4} = 0.553 = 550 \text{ mm}$**

And speed **$N = Q/Q_r D^3 = 0.18/(987 \times 10^{-6} (0.554^3)) = 1076 = 1080 \text{ RPM}$**

Because of the square and cubic terms, rounding errors can be large. Use as many decimal places as possible in your calculations and round answers to nearest 10 RPM or mm.

The Head and Discharge Ratios for the design are the same as for the model:

$$H_r = 7/(1076^2 0.553^2) = 19.8 \times 10^{-6}$$

$$Q_r = 0.18/(1076(0.553)^3) = 987 \times 10^{-6}$$

The apparent imbalance in a larger pump running at lower speed is due to the relative changes in flowrate and head. Flowrate increases by more than 4x, while head increases by less than 2x. The demand is for larger volume, but not proportionally larger power.