
Tutorial 3 Mass and Energy Conservation and Applications

Continuity (Mass Conservation)

1800 litres per minute flow through a 0.3 m pipe which later reduces to a 0.15 m pipe. Calculate the average velocities in the two pipes.

Let's convert that to SI units to begin with. $1800 \text{ litres/min} = 1800/1000/60 = 30 \times 10^{-3} \text{ m}^3/\text{s}$

Continuity allows us to calculate the average velocity from a flowrate so long as we know the flow area. A check on units confirms which way around the terms in this relationship must go:

$$[Q] = \text{m}^3/\text{s}$$

$$[V] = \text{m}/\text{s}$$

$$[A] = \text{m}^2$$

$$\text{m}^2 \times \text{m}/\text{s} = \text{m}^3/\text{s} \text{ ie } VA = Q$$

So $V = Q/A$

The flow area of a pipe is $\pi D^2/4$, so the two areas are

$$A_1 = \pi(0.3^2)/4 = 70.69 \times 10^{-3} \text{ m}^2$$

$$A_2 = \pi(0.15^2)/4 = 17.67 \times 10^{-3} \text{ m}^2$$

So $V_1 = Q/A_1 = 30 \times 10^{-3} / 70.69 \times 10^{-3} = 0.424 \text{ m/s}$

and $V_2 = Q/A_2 = 30 \times 10^{-3} / 17.67 \times 10^{-3} = 1.696 \text{ m/s} = 4V_1$

The factor of 4 comes in because when a diameter is halved, its area is quartered. In continuity, this means the velocity is quadrupled. Since the kinetic energy of the flow is proportional the velocity squared (as $KE = \frac{1}{2} mV^2$), so the halved diameter can produce a 16-fold increase in flow energy!

If the velocity in a 12 mm pipe is 0.5 m/s, what is the velocity in a 3 mm jet issuing from a nozzle attached to the pipe?

We can use the continuity relationship to derive a discharge, or we can just use what we now know about the relationship between area and velocity. Let's do it both ways as a double-check:

We have $Q = VA$

In the 12 mm pipe, $A = \pi(0.012^2)/4 = 113.1 \times 10^{-6} \text{ m}^2$

And since $V = 0.5 \text{ m/s}$, $Q = 0.5(113.1 \times 10^{-6}) = 56.55 \times 10^{-6} \text{ m}^3/\text{s}$

Then $V = Q/A$

In the 3 mm pipe, $A = \pi 0.003^2/4 = 7.07 \times 10^{-6} \text{ m}^2$

So $V = 56.55 \times 10^{-6} / 7.1 \times 10^{-6} = 7.96 \text{ m/s}$

Using the proportional relationship between area and velocity, we have a much simpler and more accurate rule to go by:

$$(D_1/D_2)^2 = (12/3)^2 = 16$$

Thus $V_2 = 16V_1 = 16(0.5) = 8 \text{ m/s}$ with no rounding error. Nice!

The open tank in Figure 1 contains water and is being filled through section 1 and 3 while draining through 2.

- Derive an expression for the water level change rate dh/dt in terms of flows Q_1 , Q_2 and Q_3 (as variables, don't specify values yet) and tank diameter d .
- If the water level h is constant, determine the exit velocity V_2 for the values $V_1 = 3 \text{ m/s}$ and $Q_3 = 0.01 \text{ m}^3/\text{s}$

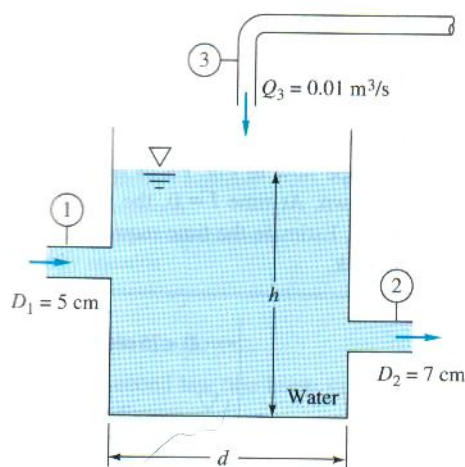


Figure 1: Tank in Q1

a) If there is any difference in the net inflow and outflow rates, the tank will fill or empty at that rate. We can work out the rate at which the volume is changing:

$$dV/dt = Q_1 + Q_3 - Q_2$$

This makes sense, because all terms are volume per time (ie the filling or emptying is a flowrate).

The height h is a function of volume,

$$h = V/A = V/(\pi d^2/4)$$

h is a function of one variable, V. To find dh/dt we equate dV/dt in the same ratio:

$$\frac{dh}{dt} = \frac{dV}{dT} / (\pi d^2/4) = (Q_1 + Q_3 - Q_2)/(\pi d^2/4)$$

Another way to look at this, is that dh/dt is a velocity (m/s) so the VA component for the tank makes up the “missing” flowrate: $Q_m = Q_1 + Q_3 - Q_2$, and $V_{\text{tank}} = dh/dt = Q_m/A_{\text{tank}}$

b) If the water level is constant, dh/dt = 0 hence $Q_1 + Q_3 - Q_2 = 0$ or $Q_2 = Q_1 + Q_3$

We are told $Q_3 = 0.01$, $V_1 = 3$ m/s and $D_1 = 2$ cm.

By continuity $Q_1 = V_1 A_1 = 3(\pi D_1^2/4) = 3(\pi 0.05^2/4) = 5.88 \times 10^{-3} \text{ m}^3/\text{s}$

So $Q_2 = Q_1 + Q_3 = 0.01 + 0.00588 = 0.01588 \text{ m}^3/\text{s}$

And $V_2 = Q_2/A_2 = Q_2/(\pi D_2^2/4) = 0.01588/(\pi 0.07^2/4) = \underline{\underline{4.12 \text{ m/s}}}$

A conventional rocket motor is operating in steady state as shown in Figure 2. The combustion products flowing out of the exhaust nozzle approximate a perfect gas with molecular mass about 28 (specific gas constant approximately $290 \text{ J kg}^{-1} \text{ K}^{-1}$) The exhaust emerges into atmospheric pressure at a measured temperature of 593°C (866 K). For the given conditions evaluate V_2 in m/s. (Hint: use mass flowrate in kg/s as gas density ρ x volume flowrate Q)

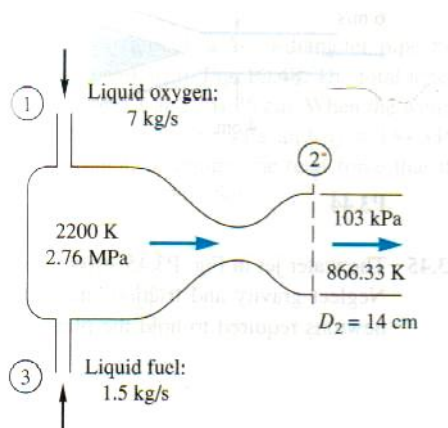


Figure 2: Rocket motor in Q2

By the conservation of mass the mass inflow and outflow rates must be equal:

$$\text{Inflow rate} = 7 + 1.5 = 8.5 \text{ kg/s} = \text{outflow rate}$$

The mass outflow rate is determined by the density and volumetric flowrate of the gas:

$$\text{Volumetric flowrate} = Q = VA$$

$$\text{Mass flowrate} = \dot{m} = \rho Q = \rho VA$$

The nozzle area is $A_2 = (\pi D_2^2/4) = (\pi 0.14^2/4) = 0.015 \text{ m}^2$

The outflow density is $\rho_2 = P/RT = 103000/(290(866)) = 0.41 \text{ kg/m}^3$

The mass flowrate $\dot{m} = \rho VA$

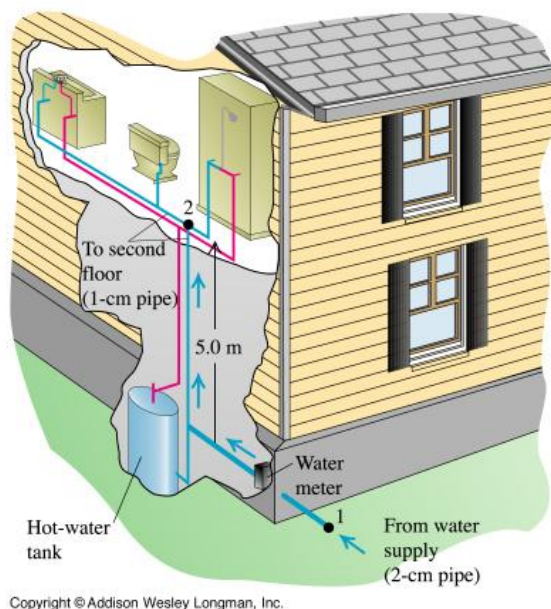
So $V_2 = \dot{m}/(\rho_2 A_2) = 8.5/(0.41(0.015)) = 1350 \text{ m/s} = 1.4 \text{ km/s}$

This high velocity is typical of rocket engines, which can eject exhaust at up to Mach 10.

Energy Conservation

Water enters a house through a pipe with an inside diameter of 2.0 cm at an absolute pressure of $4.0 \times 10^5 \text{ Pa}$ (about 4 atm). A 1.0 cm diameter pipe leads to the second floor bathroom 5.0 m above. When the flow speed at the inlet pipe is 1.5 m/s, calculate:

- a) the flow speed of the water in the bathroom
- b) the pressure of the water in the bathroom



This is an application of both continuity and the energy conservation laws embodied in the Bernoulli principle. Usually we need at least these two laws to solve fluid problems.

- a) The fluid is incompressible, so we can apply the simpler form of continuity:

At the 2 cm pipe we know $V = 1.5 \text{ m/s}$

Thus $Q = V_1 A_1 = 1.5 \pi (0.02)^2 / 4 = 471.2 \times 10^{-6} \text{ m}^3/\text{s}$

but $Q = 471.2 \times 10^{-6} = V_2 A_2$

Where the area of the bathroom pipe is $A_2 = \pi (0.01)^2 / 4 = 78.54 \times 10^{-6} \text{ m}^2$

That is $V_2 = Q/A_2 = 471.2 \times 10^{-6} / 78.54 \times 10^{-6} = 6 \text{ m/s}$

b) The Bernoulli principle provides us with a relationship between, pressure, velocity and elevation. Since we know the pressure and velocity at the inlet, and the velocity and elevation at the outlet (bathroom), we can determine the pressure in the bathroom:

$$\frac{V_1^2}{2g} + z_1 + \frac{P_1}{\rho g} + PumpHead - TurbineHead - HeadLoss = \frac{V_2^2}{2g} + z_2 + \frac{P_2}{\rho g}$$

There are no pumps or turbines, and we can assume no losses for the moment: so all those terms are zero. Also we can decide on our own datum of elevation, say at the inlet, thus $z_1 = 0$:

so
$$\frac{V_1^2}{2g} + 0 + \frac{P_1}{\rho g} + 0 - 0 - 0 = \frac{V_2^2}{2g} + z_2 + \frac{P_2}{\rho g}$$

or
$$1.5^2/19.62 + 4 \times 10^5/9800 = 6^2/19.62 + 5 + P_2/9800$$

$$0.115 + 40.82 = 6.835 + P_2/9800$$

Thus
$$P_2 = (0.115 + 40.82 - 6.835)9800 = 334.2 \times 10^3 \text{ Pa}$$

The velocity downstream of a sluice gate (Fig. 2) is assumed to be uniform. Express V_2 in terms of H and h for this inviscid flow. Use a streamline:

(a) Along the top

(b) Along the bottom

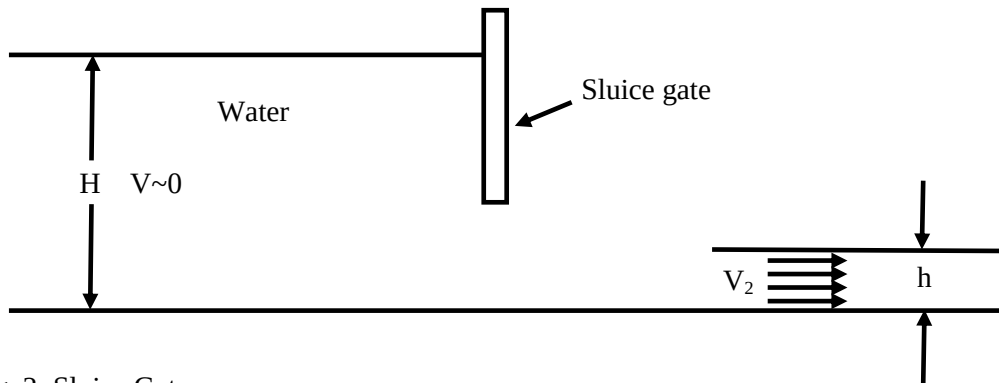


Fig. 2: Sluice Gate

This is a simple approximation of open channel flow that is well worth understanding. Using streamlines in different parts of the model is almost a trick question, since the uniform flow distribution means the flow is in effect only 1-dimensional, or is modelled as such. We should therefore find that any expression worked out along different streamlines comes out the same way every time. In fact this is exactly what it does:

a) A streamline “along the top” means we follow a point on the surface of the reservoir down the back face of the wall and onto the surface of the tailwater. We can write Bernoulli’s equation for pressure, elevation and velocity terms at the reservoir and the tailwater and try to equate V to H as required:

Again no pumps, turbines or losses so

$$\frac{V_1^2}{2g} + z_1 + \frac{P_1}{\rho g} = \frac{V_2^2}{2g} + z_2 + \frac{P_2}{\rho g}$$

At the water surface of the reservoir, $P = 0$ (gauge pressure), $V \sim 0$ (the water is deep and so flows very slowly) and $z = H$ relative to the bed:

$$\text{Thus } 0 + H + 0 = H = \frac{V_2^2}{2g} + z_2 + \frac{P_2}{\rho g}$$

At the water surface of the tailwater, $P = 0$ again, V is high and cannot be neglected, and $z = h$:

$$\text{So } H = \frac{V_2^2}{2g} + h + 0$$

$$\text{Or } V_2 = \sqrt{2g(H - h)}$$

This approximation is well borne out by observation despite the approximation of uniform flow.

b) A streamline “along the bottom” means we follow a point on the streambed from the reservoir through to the tailwater. This is where the approximation gets a little ridiculous, because we know from the no-slip condition first articulated by Newton himself that the velocity at the bed is always zero, so a streamline in reality won’t do anything. But let’s see what the approximation has for us anyway.

At the bed in the reservoir, $P = \rho gH$, $V \sim 0$ and $z = 0$ relative to our datum, the bed.

$$\text{Thus } 0 + 0 + \frac{P_1}{\rho g} = \frac{\rho gH}{\rho g} = H = \frac{V_2^2}{2g} + z_2 + \frac{P_2}{\rho g}$$

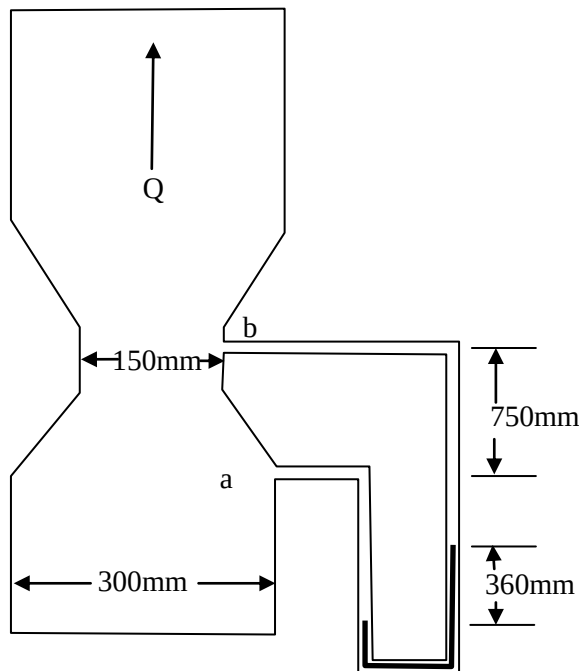
At the bed of the tailwater, $P = \rho gh$ again, V is high and cannot be neglected, and $z = 0$:

$$\text{So } H = \frac{V_2^2}{2g} + 0 + \frac{\rho gh}{\rho g} = \frac{V_2^2}{2g} + h$$

This is the same as before, with the terms rearranged as pressure and potential energy equate to the same thing.

Thus $V_2 = \sqrt{2g(H - h)}$ as a general rule for sluice gates. V is an average velocity and is an expression of an energy transformation all streamlines in the system tend to follow, on average.

Determine the vertical flow Q of water through the Venturi meter (not drawn to scale) assuming no energy is lost. What is the flow speed in the 300 mm pipe? The mercury in the gauge has a specific gravity of 13.6.



The venturi meter uses the Bernoulli principle, as the velocity increases in the throat so the pressure must decrease, as shown by the deflection of the manometer due to the higher pressure at point a compared to point b.

Let's work out the pressure difference first: we can follow the manometer line by stages from point a to point b, adding and subtracting static pressures as we go. There are three unknown depths, 1) between point a and the low mercury surface, 2) between that point and the bottom of the U, and 3) between the high mercury surface and point b. Let's call them h_1 , h_2 and h_3 for the moment:

$$P_a + \rho_w g h_1 + \rho_m g h_2 - \rho_m g h_2 - \rho_m g 0.36 - \rho_w g h_3 = P_b$$

$$\text{Or } P_a + \rho_w g (h_1 - h_3) - \rho_m g 0.36 = P_b$$

$$\text{Now the total height of the U-tube} = h_1 + h_2 + 0.75 = h_3 + 0.36 + h_2$$

$$h_2 \text{ cancels so } h_1 - h_3 = 0.36 - 0.75 = -0.39$$

$$\text{And } P_a - \rho_w g 0.39 - \rho_m g 0.36 = P_b$$

$$\text{Or } P_a - P_b = 0.39 \rho_w g + 0.36 \rho_m g = 0.39(1000)9.8 + 0.36(13600)9.8 = 51.8 \times 10^3 \text{ Pa}$$

Now we can write Bernoulli's equation, with no losses:

$$\frac{V_a^2}{2g} + z_a + \frac{P_a}{\rho g} = \frac{V_b^2}{2g} + z_b + \frac{P_b}{\rho g}$$

We know $P_a - P_b$ so let's get it in those terms:

$$\frac{P_a - P_b}{\rho g} = \frac{V_b^2}{2g} + z_b - \frac{V_a^2}{2g} - z_a$$

$z_b - z_a$ is given as 750 mm, so

$$\frac{51.8 \times 10^3}{\rho g} = \frac{V_b^2 - V_a^2}{2g} + 0.75$$

We now have an expression for $V_b^2 - V_a^2$. We can express it all in terms of V_a by converting V_b with the area proportionality of the continuity law:

$$(D_a/D_b)^2 = (300/150)^2 = 4$$

Thus $V_b = 4V_a$

$$\text{So } \frac{51.8 \times 10^3}{\rho g} = \frac{16V_a^2 - V_a^2}{2g} + 0.75$$

$$\text{Or } \frac{51.8 \times 10^3}{\rho g} - 0.75 = \frac{15V_a^2}{2g}$$

The density referred to here is that of the water, not of the mercury (don't confuse your system parameters with your instrument parameters), so

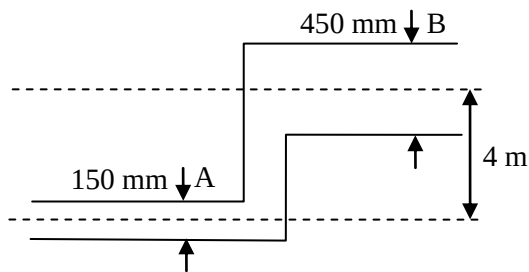
$$\frac{51.8 \times 10^3}{9800} - 0.75 = \frac{15V_a^2}{19.62}$$

$$\text{And } V_a = \sqrt{\frac{19.62}{15} \left(\frac{51.8 \times 10^3}{9800} - 0.75 \right)} = 2.483 \text{ m/s}$$

The flowrate $Q = VA = 2.483\pi(0.3^2/4) = 0.176 \text{ m}^3/\text{s}$

A pipe carrying oil of relative density 0.877 changes in size from 150 mm at section A to 450 mm at section B. Section A is 4 m lower than B. The gauge pressures at A and B are 0.9×10^5 Pa and 0.6×10^5 Pa respectively. If the flowrate is $0.15 \text{ m}^3/\text{s}$ determine the lost head and the direction of flow.

It always helps to draw a diagram (not necessarily to scale):



The flowrate and diameters allow us to calculate a velocity for A and B:

$$Q = VA$$

$$V = Q/A$$

$$V_A = 0.15/(\pi 0.15^2/4) = 8.488 \text{ m/s}$$

$$V_B = 0.15/(\pi 0.45^2/4) = 0.943 \text{ m/s}$$

Without knowing the direction of the flow or the magnitude of losses and we can write the Bernoulli equation as:

$$\frac{V_1^2}{2g} + z_1 + \frac{P_1}{\rho g} = \frac{V_2^2}{2g} + z_2 + \frac{P_2}{\rho g}$$

Let's solve the two sides of the equation and compare them:

$$\text{Section A: } 8.488^2/19.62 + 0 + 0.9 \times 10^5/(877 \times 9.8) = 14.14 \text{ m}$$

$$\text{Section B: } 0.943^2/19.62 + 4 + 0.6 \times 10^5/(877 \times 9.8) = 11.03 \text{ m}$$

The difference between the two values is the lost head:

$$h_l = 14.14 - 11.03 = 3.11 \text{ m}$$

We can now write the equation more fully and locate the losses:

$$\frac{V_1^2}{2g} + z_1 + \frac{P_1}{\rho g} - \text{HeadLoss} = \frac{V_2^2}{2g} + z_2 + \frac{P_2}{\rho g}$$

This form of the equation assumes flow from section 1 to section 2, with head loss a positive amount. The equivalent formulation for this situation is thus:

$$8.488^2/19.62 + 0 + 0.9 \times 10^5/(877 \times 9.8) - 3.11 = 0.943^2/19.62 + 4 + 0.6 \times 10^5/(877 \times 9.8)$$

Which indicates, by the position of the losses needed to equate the two sections, that the flow is from A to B.

A 150 mm pipe 180 m long raises water by a height of 20 m. The frictional shear stress between the fluid and walls is 15 Pa. Determine the pressure change in the pipe and the lost head in metres

Start by writing Bernoulli's equation for points A and B:

$$\frac{V_A^2}{2g} + z_A + \frac{P_A}{\rho g} - \text{HeadLoss} = \frac{V_B^2}{2g} + z_B + \frac{P_B}{\rho g}$$

We need to find the pressure difference:

$$\frac{V_A^2}{2g} - \frac{V_B^2}{2g} + z_A - z_B - \text{HeadLoss} = \frac{P_B}{\rho g} - \frac{P_A}{\rho g}$$

Or

$$P_B - P_A = \rho g \left(\frac{V_A^2}{2g} - \frac{V_B^2}{2g} + z_A - z_B - \text{HeadLoss} \right)$$

The pipe remains at the same diameter, so there are no velocity changes: $V_A = V_B$:

$$P_B - P_A = \rho g \left(z_A - z_B - \text{HeadLoss} \right)$$

The change in elevation is known: Z_A is 20m lower than Z_B so

$$Z_A - Z_B = -20 \text{ m}$$

A headloss is created by the shear stress on the walls. Shear stress is a form of pressure, which can be equated to headloss by working out the shear force along 180 m of 150 mm pipe:

$$\text{Pipe circumference } C = \pi D = 0.15\pi = 0.471 \text{ m}$$

$$\text{Pipe friction surface area} = CL = 0.471(1500) = 84.82 \text{ m}^2$$

$$\text{Shear Force } F_t = \tau A = 15(84.82) = 1.27 \text{ kN}$$

This force is equivalent to a net back pressure on the pipeline's end face:

$$F_t = P_b \pi D^2 / 4 = P_b \pi (0.15)^2 / 4 = P_b (0.0177)$$

The net back pressure is thus:

$$P_b = F_t / 0.0177 = 1.27 \times 10^3 / 0.0177 = 71.9 \text{ kPa}$$

Which is equivalent to a headloss: $H_b = P_b / \rho g = 71.9 \times 10^3 / 9800 = 7.3 \text{ m}$

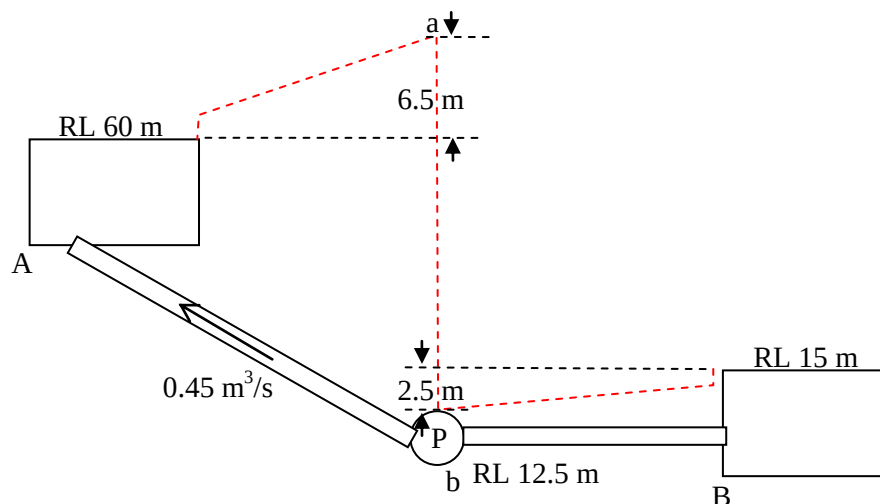
So we have:

$$P_B - P_A = \rho g \left[z_A - z_B - \text{HeadLoss} \right] = 9800(-20 - 7.3) = \underline{-267.5 \text{ kPa}}$$

Water at 32.2 °C is to be pumped from a river at a velocity of 2 m/s. What is the maximum elevation of the pump above the water surface before the water column breaks due to low-temperature boiling? Use atmospheric pressure = 1.013×10^5 Pa. The vapour pressure of water at this temperature is 4800 Pa. The inlet pipe has a known loss factor of 3.0

The two water reservoirs in a large town are connected to a pumping station. The water level in primary reservoir A is 60 m above sea level, while that of secondary reservoir B is at 15 m. The main pump in the station is at elevation 12.5 m. The pump must transfer $0.45 \text{ m}^3/\text{s}$ from the secondary to the primary reservoir, at which rate the known head losses are 2.5 m from reservoir B to the pump and 6.5 m from the pump to reservoir A. How much fluid power must the pump add to the system? Plot the energy line of the system.

Again let's start with a diagram, not necessarily to scale:



The dotted line is a sketch of the total energy line or TEL of the system. Minor entry and exit losses appear as small steps at the two reservoirs A and B. Pipe losses appear as sloping sections of the TEL. The pump's energy input appears as a major step upwards between points b and a. The size of this step, in metres of head, can be extracted from the diagram: Working from left to right, adding and subtracting the known and unknown values of head, we have:

$$60 + 6.5 - \text{pump head} + 2.5 = 15$$

So pump head = $60 + 6.5 + 2.5 - 15 = 54$ m

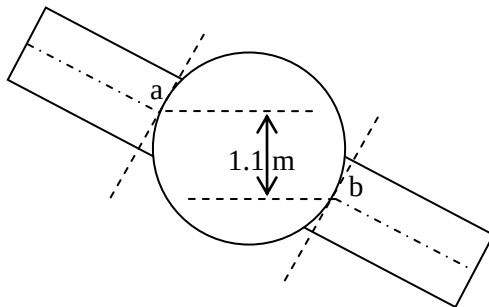
From p72 (study guide) we can convert this to fluid power:

$$Power = \rho g Q \Delta H$$

$$= 1000(9.8)0.45(54) = 238.1 \times 10^3 \text{ Watts}$$

Water flows through a turbine at a rate of $0.2 \text{ m}^3/\text{s}$, falling 1.1 m as it passes through the turbine. The gauge pressures upstream and downstream of the turbine are 1.5×10^5 and -0.3×10^5 respectively. What power is extracted from the water by the turbine? If on another day the turbine power is 50 kW for the same pressure losses, and assuming the turbine is 100% efficient, what is the flowrate through the turbine?

Draw a diagram:



Write Bernoulli's equation between sections a and b with turbine losses:

$$\frac{V_a^2}{2g} + z_a + \frac{P_a}{\rho g} + PumpHead - TurbineHead - HeadLoss = \frac{V_b^2}{2g} + z_b + \frac{P_b}{\rho g}$$

We can assume that $V_a = V_b$ (no change in pipe diameter is referenced) so those terms cancel;

There is no pump head and we can incorporate headlosses as part of an inefficient turbine head;

With point a as a datum we have $z_a = 0$ and $z_b = -1.1$ m;

The equation thus reduces to:

$$0 + 1.5 \times 10^5 / 9800 - \text{Turbine Head} = -1.1 - 0.3 \times 10^5 / 9800$$

$$\text{Turbine Head} = 1.5 \times 10^5 / 9800 + 1.1 + 0.3 \times 10^5 / 9800 = 19.47 \text{ m}$$

This represents a fluid power of $Power = \rho g Q \Delta H = 9800(0.2)19.47 = 38.16 \times 10^3 \text{ W}$

Note that this includes headlosses due to friction and turbulence in the turbine, usually represented as the inefficiency of the turbine. The actual electric power produced by the turbine is less than the fluid power extracted.

Assuming 100% efficiency and constant pressure losses at different operating speeds (both of which are unrealistic), the turbine power of 50 kW represents a flowrate of:

$$50 \times 10^3 = 9800Q(19.47)$$

$$Q = 50 \times 10^3 / (9800 \times 19.47) = 0.262 \text{ m}^3/\text{s}$$

A 16 mm diameter nozzle water tap delivers 1.2 litres per minute into a sink of unspecified depth z_0 (Figure 3).

- Assuming an initial elevation of z_0 for the tap nozzle = the depth of the sink, develop an expression for the velocity of the flow as a function of depth h it falls under gravity. You don't need the Bernoulli or continuity principle. Ignore wind resistance and any other losses.
- Use the Bernoulli principle or your own reasoning to describe the pressure in the flow as a function of h . Does the gauge pressure increase, decrease or remain the same as it falls?
- How would velocity and pressure vary if the flow was into a vertical *frictional* pipe of constant diameter? Do no calculations, describe qualitatively.

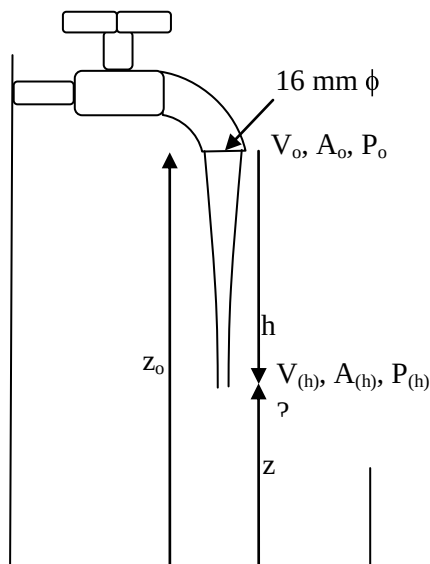


Figure 3: Hypothetical water tap (Q3)

a) Convert to SI units: $Q = 1.2 \text{ (litres/minute)} = 1.2 \times (1 \times 10^{-3} \text{ litres/m}^3) / (60 \text{ secs/minute}) = 2 \times 10^{-5} \text{ m}^3/\text{s}$

Determine initial velocity: $V = Q/A$

$$A = \text{nozzle area} = \pi D^2/4 = \pi(16 \times 10^{-3})^2/4 = 2 \times 10^{-4} \text{ m}^2$$

$$V_o = Q/A = 2 \times 10^{-5} / 2 \times 10^{-4} = 0.1 \text{ m/s}$$

Consider gravitational acceleration: from first year physics we can express velocity under acceleration in terms of time *or* distance:

$$V = V_o + gt \text{ or}$$

$V^2 = V_o^2 + 2gh$ where t is the time and h is the distance travelled in fall.

Thus $V = \sqrt{(V_o^2 + 2gh)} = \underline{(0.01 + 19.6h)^{1/2}}$

b) The Bernoulli Principle (no losses):

$$P + \rho V^2/2 + \rho gz = P_o + \rho V_o^2/2 + \rho gz_o$$

The initial pressure is atmospheric, ie the water emerges in equilibrium with the atmosphere. This is because the stream is not isolated from the atmosphere and there are no lateral accelerations:

$$P_{\text{gauge}} = P_o - \text{atmospheric} = 0 \text{ Pa}$$

Given the other initial values, the equation becomes:

$$P + \rho V^2/2 + \rho gz = 0 + 1000(0.1)^2/2 + 9800z_o = 5 + 9800z_o$$

Or $P = P_{\text{gauge}} = 5 + 9800z_o - \rho V^2/2 - \rho gz$

We have an expression for V as a function of h (part a). The elevation z is also a function of depth h ,

$$z = z_o - h$$

Thus $P_{\text{gauge}} = 5 + 9800z_o - \rho V^2/2 - 9800(z_o - h)$

$$= 5 - 1000V^2/2 + 9800h$$

$$= 5 - 1000((0.01 + 19.6h)^{1/2})^2/2 + 9800h$$

$$= 5 - 1000(0.01 + 19.6h)/2 + 9800h$$

$$= \underline{5 - 10/2 - 9800h + 9800h = 0}$$

Gauge Pressure as function of depth (Pa)

The gauge pressure is a constant zero. As the water falls, all its potential energy is converted to kinetic, and there is no change in the component of energy that is pressure.

c) When friction is involved, as in a downhill pipe flow, energy is lost to the walls of the pipe, so pressure *falls* with distance travelled and velocities are constant in proportion to pipe area. If the downhill flow is very slow and empties under the standing water level (say a pipe into a half-full sink) the pressure may increase with downhill distance as the situation approximates static.

In a realistic free fall situation, air friction drags the air around the stream into a flow at a velocity V and hence lower pressure P similar to the reduced pressure in the stream. At fairly low velocities a flow falling through air will break up into droplets due to turbulent instabilities over this exchange of momentum. Liquids cannot fall far in a continuous unconfined stream.

A section in a pipe flow, called a venturi, develops a low throat pressure that can aspirate fluid upward from a reservoir. Using Bernoulli's equation with no losses, derive an expression for the flowrate Q in terms of the height h and the diameters D_1 and D_2 that is just sufficient to bring reservoir fluid into the throat.

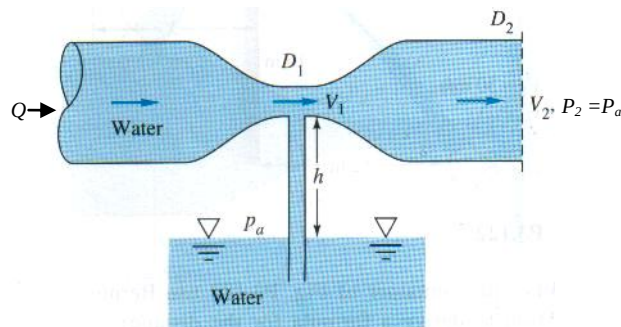


Figure 4: Venturi

Bernoulli's equation:

$$P_1 + \rho V_1^2/2 + \rho g z_1 = P_2 + \rho V_2^2/2 + \rho g z_2$$

Elevations are the same, and P_2 is given as atmospheric so the gauge pressure there is zero:

$$P_1 + \rho V_1^2/2 = 0 + \rho V_2^2/2$$

Or
$$P_1 = \rho(V_2^2 - V_1^2)/2$$

To create a suction, this must be negative. To achieve a suction of height h , the pressure must be at least $P_1 = -\rho gh$

So
$$-\rho gh = \rho(V_2^2 - V_1^2)/2$$

V_1 must be larger than V_2 . This can be arranged by the ratio of throat to pipe diameters.

Since
$$V_1 A_1 = V_2 A_2$$

$$V_2 = V_1 A_1/A_2 = V_1 D_1^2/D_2^2 \text{ since } A \sim D^2$$

So
$$-\rho gh = \rho (V_1^2 (D_1/D_2)^4 - V_1^2)/2 = \rho V_1^2 ((D_1/D_2)^4 - 1)/2$$

Or
$$V_1 = (2gh/(1 - (D_1/D_2)^4))^{1/2} \quad \text{Note that density cancels and 2 cross-multiplies}$$

The flowrate $Q = VA = V_1 \pi D_1^2/4$

Thus
$$Q = \pi D_1^2/4 \times (2gh/(1 - (D_1/D_2)^4))^{1/2}$$

Venturi flowrate m³/s

Gas Dynamics

What diameter of pipe is necessary to carry 0.25 kg/s of air with a velocity of 6 m/s? The air is at 27 °C and the absolute pressure is 2.3×10^5 Pa. R (Air) = $287 \text{ J kg}^{-1} \text{ K}^{-1}$

The density of the air is $\rho = P/RT = 2.3 \times 10^5 / (287(300)) = 2.67 \text{ kg/m}^3$

We know the mass flowrate and velocity: These are related by:

$$\dot{m} = \rho Q = \rho VA = 2.67(6)A = 0.25 \text{ kg/s}$$

$$\text{Thus } A = \dot{m} / \rho V = 0.25 / (2.67(6)) = 0.016 \text{ m}^2 = \pi D^2 / 4$$

$$\text{And } D = \sqrt{(4(0.016)/\pi)} = \underline{0.14 \text{ m}}$$

A 200 mm pipe carries air at average velocity 24.4 m/s, 148 KPa absolute pressure and 27°C. The gas constant for air is $R = 287.1 \text{ J/kg/K}$. (a) What is the mass flow rate in Kg/s? (b) The pipe reduces to 100 mm and the pressure and temperature change to 131 KPa and 11°C. What is the flow velocity? (c) Compare the volume flow rates in m³/s between the two pipes. Should they be the same? Which flowrate, the mass or volume flowrate, is the same? Why?

Worked solution:

(a) We need the density of air at the specified pressures and temperatures. We have

$$P = \rho RT \text{ with } T \text{ in Kelvin}$$

$$\text{Thus } \rho = P/RT = 148 \times 10^3 / (287.1(27+273)) = 1.718 \text{ kg/m}^3$$

The volumetric flowrate is

$$Q = VA = 24.4(\pi 0.2^2 / 4) = 0.767 \text{ m}^3/\text{s}$$

The mass flowrate is just the volume flowrate multiplied by the density:

$$Q_m = \rho Q = 1.718(0.767) = \underline{1.316 \text{ Kg/s}}$$

(b) In steady state the mass flowrate is constant along the pipe as it varies in diameter. This is a consequence of the conservation of matter, as mass flows out of the 200 mm pipe the same mass flows into the 100 mm pipe. To determine the velocity we need the volume flowrate, which varies as the density varies:

$$\text{Again } \rho = P/RT = 131 \times 10^3 / (287.1(11+273)) = 1.607 \text{ kg/m}^3$$

Then given $Q_m = 1.316 \text{ Kg/s} = \rho Q$ we have

$$Q_2 = 1.316 / \rho = 1.316 / 1.607 = 0.819 \text{ m}^3/\text{s}$$

By the continuity equation we have $Q = VA$

Thus $V_2 = Q_2/A_2 = 0.819/(\pi 0.1^2/4) = \underline{104.3 \text{ m/s}}$

- (c) The volume flowrates are 0.767 and 0.819 m³/s respectively. These are not the same because the gas decompresses as it moves into the narrower pipe. The volume flowrate depends on the density, which is reliably constant only for liquids. The mass flowrate, as discussed above, is constant for steady state systems. In unsteady flows the amount of matter entering and moving through a system varies in time and space.