

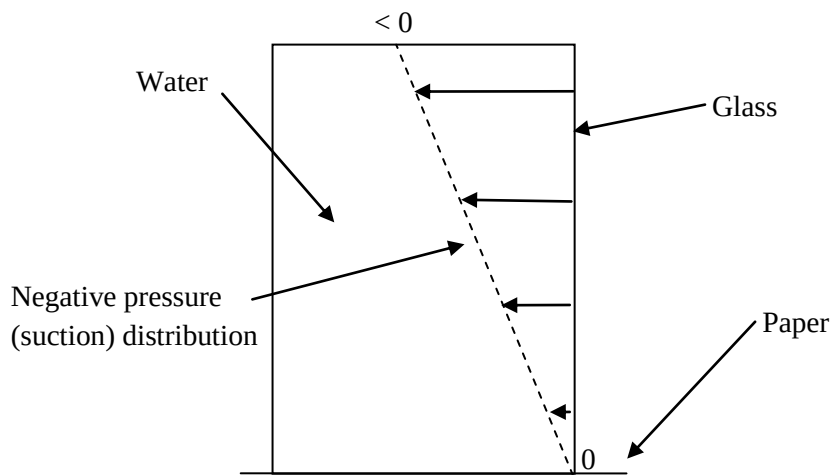
Pressure and Density

A 12 cm high glass is completely filled with water. A sheet of paper is placed over the top, and the glass and paper are carefully inverted (try it yourself). Calculate the pressure at the top and bottom of the glass (inverted) and explain why the water does not fall out. How tall would the glass need to be for water to fall out, assuming a vapour pressure of 2500 Pa at 20 °C ?

Although wet paper will stick to glass, the adhesion is not strong enough to support even this small mass of water, so the paper cannot be under any kind of positive pressure from within the glass. There must be atmospheric pressure on both the air side and the water side of the paper. Recall that this is zero gauge pressure.

So far so good. The question remains, what is supporting the water, since the water body has mass and is pulled down by gravity? The resulting pressure distribution must still increase with depth in the downward direction as a result of the overlying burden. So the pressure at the top of the water column (ie the inverted bottom of the glass) must be lower than the pressure at the covering paper. But if the pressure at the covering paper is zero, then the inverted bottom must be under negative pressure, or suction.

The resulting pressure profile is linear, as per any static pressure distribution, with negative values everywhere:



The water column is thus in suction or tension throughout. It will stretch by a tiny amount, but not enough to effect the paper's contact with the glass rim. No air can get in so long as no water comes out, and the water cannot come out because any uniform movement downward would cause the suction to increase. The paper creates a constraint that the initiation of such a movement has to be uniform, so it simply cannot happen. The only way to break the water column is to pull away the paper.

Note that if there is any air in the glass to begin with, the air will stretch by a discernible amount, allowing the paper to separate from the rim. This lets more air in, breaking the tension and so releasing the water column from its unlikely container.

Clean up the mess, time for the maths.

Ian Stewart

The negative pressure at the inverted bottom of the glass is an ordinary static pressure with a *negative* depth:

$$P(\text{gauge}) = 0 - \rho g H = -1000(9.8)(0.12) = -1176 \text{ Pa}$$

In terms of absolute pressure, the pressure includes the atmospheric pressure, which depends on the weather (as a variable) and the local altitude. Let's say we're at sea level at average pressure:

$$P(\text{absolute}) = 101300 - \rho g H = 101300 - 1000(9.8)(0.12) = 100.1 \text{ kPa}$$

The water in the glass is in a state of chemical equilibrium in which some molecules of water are present as vapour, but are prevented from accumulating as gas bubbles by the overlying air pressure. Placing a water column in suction risks reducing the pressure to the point where the water will effectively boil at this temperature, forming pure water vapour bubbles within the liquid. In an inverted glass, this has the same effect as allowing air in: the vapour will stretch, breaking the seal and allowing the container to let more air in.

This occurs when the pressure falls to the value of the vapour pressure. To work with chemical equilibria we must express pressures (and temperatures) in absolute terms:

$$P_{\text{vap}} = 2500 \text{ Pa} = 101300 - \rho g H$$

$$\text{Thus } H = (101300 - 2500) / (9.8 \times 1000) = 10.08 \text{ m}$$

ie the glass must be 10 m tall before this experiment will fail. This is in fact the height of a water barometer, a rarely constructed device. Mercury is used because it is far more dense and has a very small vapour pressure:

$$H(\text{mercury barometer}) = (101300) / (9.8 \times 13600) = 0.760 \text{ m}$$

As in 760 mm of Mercury, the barometric standard atmospheric pressure.

A weather balloon with a volume of 10 m^3 is filled with helium at 20°C and 200 kPa absolute pressure. Determine the mass (in kg) of helium in the balloon. Repeat for a pressure of 100 kPa absolute. Repeat again for volumes of 5 m^3 and 15 m^3 , at pressures of 100 and 200 kPa . Plot a graph of mass of helium against balloon diameter for both pressure cases.

We can use the ideal gas law $PV = nRT$ or its formulation in terms of density $P = \rho RT$ to relate pressure to temperature and amount of gas:

Specific Gas Constant **R** (Helium) = 2077 J/kg/K (Table page 15 Study Guide)

$$P_1 = \rho_1 RT$$

$$\rho_1 = P_1 / RT = 200 \times 10^3 / (2077 \times 293) = 0.329 \text{ kg/m}^3$$

$$M_1 = \rho_1 \forall$$

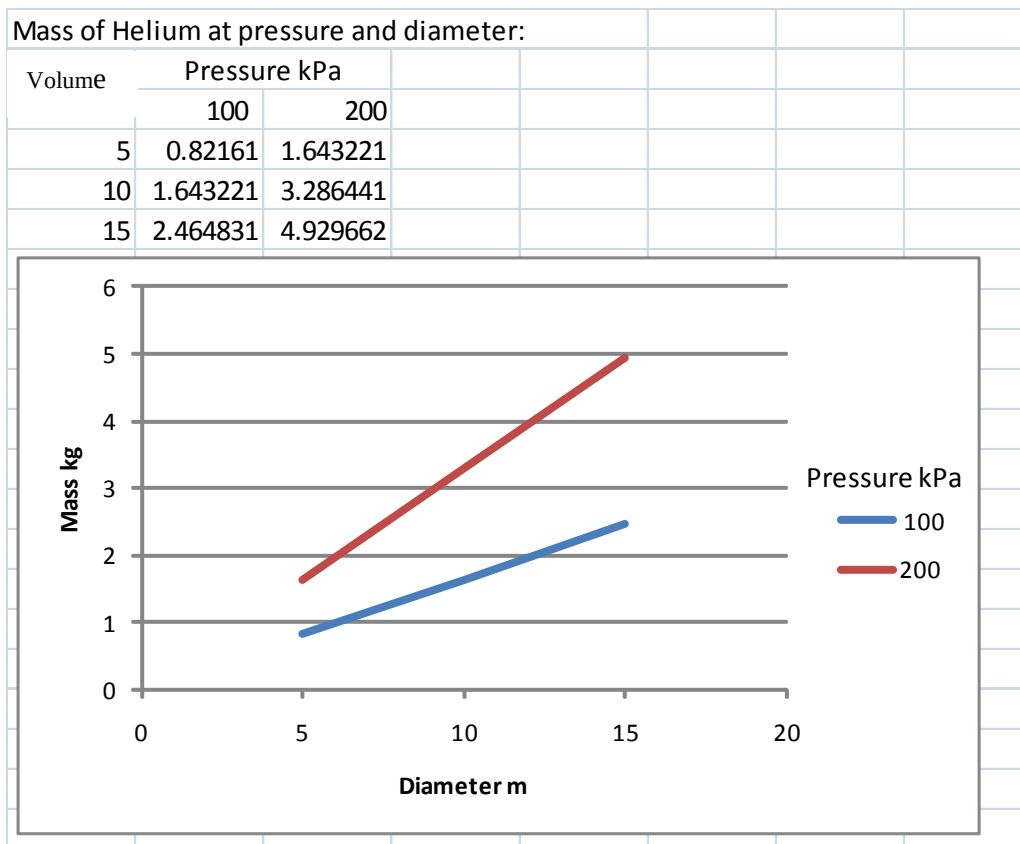
$$= 0.329 (10) = 3.29 \text{ kg}$$

$$\rho_2 = P_2 / RT = 100 \times 10^3 / (2077 \times 293) = 0.164 \text{ kg/m}^3$$

$$M_2 = \rho_2 \forall$$

$$= 0.164 (10) = 1.64 \text{ kg}$$

The results can be efficiently compared and plotted using a spreadsheet:



The density of atmospheric air varies with elevation, decreasing with increasing altitude. Using the following data table, estimate the total mass of the Earth's atmosphere. Assume that the Earth is a perfect sphere of radius 6377 km and the atmosphere is a perfect spherical shell 100 km thick. What approximations can you use for the unknown data between 25 and 100 km?

Altitude m	Density kg/m ³
0	1.225
1000	1.112
2000	1.007
3000	0.9093
4000	0.8194
5000	0.7364
6000	0.6601
8000	0.5258
10000	0.4135
15000	0.1948
20000	0.08891
25000	0.04008

This is a useful exercise in numerical integration, which is an approximate method for calculating extensive properties requiring integration where the formula is either unknown or cannot be directly integrated, but where data that can be “added up” is available. The formula relating the extensive properties of mass and volume (which are meaningful only over non-zero regions of space) with the intensive property of density (which can be defined at a point) is:

$$M = \rho \forall$$

Where M is the mass and $\forall$ is the volume of an idealised spherical shell between two radii. Given a start radius of exactly 6.377×10^6 m and assuming the density in each shell is the average of data points at the bottom and top of the shell (a reasonable and simple assumption), we can calculate the volume and hence mass within each shell.

For the first shell:

$$\rho = (1.225 + 1.112)/2 = 1.169 \text{ kg/m}^3 \text{ (average of observed densities between 0 and 1000m)}$$

$$\forall = \frac{4}{3} \pi R_2^3 - \frac{4}{3} \pi R_1^3 = 4.189 ((6378 \times 10^3)^3 - (6377 \times 10^3)^3) = 511.1 \times 10^{15} \text{ m}^3$$

$$\text{And the mass is } M = \rho \forall = 1.169(511.1 \times 10^{15}) = 597.2 \times 10^{15} \text{ kg}$$

The total mass of the atmosphere can be numerically integrated by adding the shell masses together. This is easily performed with a spreadsheet:

Altitude m	Radius m	Density kg/m ³	Average Density kg/m ³	Shell Volume m ³	Shell Mass kg
0	6377000	1.225	1.1685	5.11105E+17	5.97227E+17
1000	6378000	1.112	1.0595	5.11266E+17	5.41686E+17
2000	6379000	1.007	0.95815	5.11426E+17	4.90023E+17
3000	6380000	0.9093	0.86435	5.11586E+17	4.4219E+17
4000	6381000	0.8194	0.7779	5.11747E+17	3.98088E+17
5000	6382000	0.7364	0.69825	5.11907E+17	3.57439E+17
6000	6383000	0.6601	0.59295	1.0243E+18	6.07356E+17
8000	6385000	0.5258	0.46965	1.02494E+18	4.81362E+17
10000	6387000	0.4135	0.30415	2.56515E+18	7.80191E+17
15000	6392000	0.1948	0.141855	2.56917E+18	3.64449E+17
20000	6397000	0.08891	0.064495	2.57319E+18	1.65958E+17
25000	6402000	0.04008		Total Mass to 25000 m	5.22597E+18

This numerical integration is only accurate to the vertical resolution of the data. We can use 100 km as the standard top of the atmosphere (ie where “space” starts), ie the density falls to zero at 100 km and the remaining mass is given by the same approximate formula:

$$\rho = (0.04008 + 0.00000)/2 = 0.02004 \text{ kg/m}^3$$

$$\forall = 4.189 ((6477 \times 10^3)^3 - (6402 \times 10^3)^3) = 39.1 \times 10^{18} \text{ m}^3$$

$$M = 0.02004 \times 39.1 \times 10^{18} \text{ m}^3 = 783.2 \times 10^{15} \text{ kg}$$

And the total (estimated) mass is 6.01×10^{18} kg. This is only one millionth of the total mass of the earth!

A hydrogen-filled balloon is to expand to a sphere 20 m in diameter at a height of 30 km where the absolute pressure is 1100 Pa and the temperature -40°C . If there is to be no stress in the fabric of the balloon what volume of hydrogen must be added at ground level where the absolute pressure is 101.3 kPa and the temperature 15°C ?

“No stress in the fabric” means the balloon only just expands to its full volume at 20 m diameter, and there is no pressure imbalance between inside and outside the balloon. This occurs where the internal and external pressures equalise at 1100 Pa. We can also assume thermal equilibrium, ie when the balloon and its gas content have cooled to -40°C or 233 K.

So the situation at 30 km is summarised by the ideal gas law with density as an unknown:

$$P = \rho \mathbf{R} T \quad \text{where } \mathbf{R} \text{ (hydrogen)} = 4124 \text{ J/kgK (Table page 15 Study Guide)}$$

$$\text{Hence } \rho = 1100/(4124 \times 233) = 0.001145 \text{ kg/m}^3$$

And the mass of gas can be determined by volume at the fully inflated 20 m diameter (10 m radius):

$$M = \rho V = 0.001145 \times 4\pi/3 \times 10^3 = 4.795 \text{ kg}$$

We can also work out the density of hydrogen at ground level where $T = 288 \text{ K}$:

$$\rho = 101300/(4124 \times 288) = 0.08529 \text{ kg/m}^3$$

And the volume is just

$$V = M / \rho = 4.795/0.08529 = 56.3 \text{ m}^3$$

Which, as a check, would amount to a sphere 4.75 m in diameter. In the image below, there is room for the gas to expand into the balloon's sea-level volume.



A cylinder contains 0.35 m^3 of air ($\gamma = 1.4$) at 50°C (323K) and 276 kPa absolute pressure. The air is compressed to 0.071 m^3 . (a) Assuming isothermal conditions (constant temperature), what is the pressure at the new volume? (b) Assuming adiabatic (or isentropic) conditions, what is the pressure at the new volume?

a) Isothermal conditions are not a normal, natural occurrence but must be specifically engineered in place by external cooling or heating. The maths are relatively simple though: we can use the ideal gas law with density as an unknown:

$$P = \rho RT \quad \text{where } R \text{ (air)} = 287 \text{ J/kgK (Table page 15 Study Guide)}$$

$$\text{Thus } \rho_1 = 276000/(287 \times 323) = 2.977 \text{ kg/m}^3 \text{ initially}$$

$$\text{And } M = \rho V = 2.977 \times 0.35 = 1.042 \text{ kg}$$

$$\text{So } \rho_2 = M/V_2 = 1.042/(0.071) = 14.68 \text{ kg/m}^3 \text{ after compression}$$

The temperature at the new (unknown) pressure is unchanged so we can use the ideal gas law directly for pressure:

$$P = \rho RT = 14.68(287)323 = 1.361 \times 10^6 \text{ Pa} = 1.361 \text{ MPa}$$

a) Adiabatic or isentropic conditions are a more normal occurrence in natural processes, with a slightly more complex formula in addition to the universal gas law:

$$P/\rho^\gamma = \text{constant}$$

Here γ is the ratio of specific heats, $\gamma = 1.4$ for most diatomic gases

We already know the densities at P_1 and P_2 from part a): $\rho_1 = 2.977$ and $\rho_2 = 14.68 \text{ kg/m}^3$, so

$$P/\rho^\gamma = 276000/2.977^{1.4} = 59,930 \text{ (constant)} = P_2/14.68^{1.4}$$

$$\text{Thus } P_2 = 59,930(14.68^{1.4}) = 2.577 \times 10^6 \text{ Pa} = 2.577 \text{ MPa}$$

The pressure in the adiabatic condition is nearly twice that of the isothermal condition. This is because as the pressure rises, the temperature rises to maintain a constant energy content: The new temperature can be found by the universal gas law:

$$P_2 = \rho_2 RT_2 : T_2 = P_2/\rho_2 R = 2577000/(14.68 \times 287) = 611.7 \text{ K} = 338.8^\circ \text{C}.$$

The engineering of isothermal conditions requires us to counter this temperature rise by active heat extraction (cooling).

Barometric readings at the top and bottom of a building are 730 mm and 755 mm Hg respectively. The density of Mercury is approximately 13600 kg/m^3 . Calculate the approximate height of the building. Assume an average air density of 1.2 kg/m^3 .

Let's convert everything to SI units to start with: 1 m of mercury exerts a static pressure of $\rho g(1) = 13600(9.8) = 133.3 \times 10^3 \text{ Pa}$. The two pressures stated are thus $730 \text{ mmHg} = 0.73(133.3 \times 10^3) = 97.29 \times 10^3 \text{ Pa}$ and $755 \text{ mmHg} = 0.755(133.2 \times 10^3) = 100.6 \times 10^3 \text{ Pa}$. The pressure difference between the top and bottom of the building is thus $100.6 - 97.29 = 3.33 \times 10^3 \text{ Pa}$.

Assuming a constant density of 1.2 kg/m^3 makes it a fairly simple problem: We can derive the static pressure difference ΔP over a height difference ΔH as

$$\Delta P = \rho g \Delta H$$

where ρ is 1.2 kg/m^3 , the density of the air, not the mercury (be careful to distinguish instrumental parameters such as the density of mercury from environmental parameters such as the density of air)

$$\text{Thus } \Delta H = \Delta P/\rho g = 3.33 \times 10^3/(1.2 \times 9.8) = 283.3 \text{ m}$$

This is quite a tall building at around 130 floors. Generally the small pressure differences at ground-based altitudes are not easy to detect.

Eight kilometres below the surface of the ocean the gauge pressure is 81.7 MPa. Determine the density of sea-water at this depth if the density at the surface is 1025 kg/m³ and the average bulk modulus of elasticity is 2.34 GPa.

The bulk modulus relates pressure changes to volume changes, which in turn relate to changes in density:

$$K = -\frac{\Delta P}{\Delta \forall / \forall_0} \quad (\text{Units N/m}^2 \text{ or Pa})$$

$$\text{and} \quad \rho = \frac{m}{\forall} = \frac{m}{\forall_0 + \Delta \forall} = \frac{m}{\forall_0 - \frac{\Delta P}{K} \forall_0} = \frac{m}{\forall_0} \left(\frac{1}{1 - \frac{\Delta P}{K}} \right) = \rho_0 \left(\frac{1}{1 - \frac{\Delta P}{K}} \right)$$

To find the density at a given pressure we need only apply the known pressure and modulus. The surface density is at atmospheric pressure, and any changes in pressure are relative to that, so we are safe to use the gauge pressure since that excludes the atmospheric contribution:

$$\rho = \frac{\rho_0}{1 - \frac{\Delta P}{K}} = 1025 / (1 - 81.7 \times 10^6 / 2.34 \times 10^9) = 1062 \text{ kg/m}^3$$

Calculate the pressure in the ocean at a depth of 2000 m assuming that the salt water is a) incompressible with a density of 1002 kg/m³ and b) compressible with a bulk modulus of 2.05 x10⁹ N/m² and a density at the surface of 1002 kg/m³.

- a) Incompressible means not only constant volume, but constant density, so the simple formula based on static pressure with constant density can be applied:

$$P = \rho g H \text{ where } \rho = 1002 \text{ kg/m}^3.$$

Thus at H = 2000 m we have

$$P = 1002(9.8)2000 = 19.64 \times 10^6 \text{ Pa} = 19.64 \text{ MPa, nearly 200 atmospheres.}$$

- b) This question leads to a useful exercise in *iteration*, a numerical procedure whereby we start with an initial guess and repeat or *iterate* a calculation until the improving guess converges to a solution. Iterative procedures are often useful when we cannot reduce a formula to isolate the unknowns, say for example when the unknowns appear on both sides of the equation, a common source of undergraduate frustration. Bulk modulus problems are an example of this type of problem because density and pressure are both functions of depth.

To see this, recall that the bulk modulus relates pressure changes to volume changes:

$$K = -\frac{\Delta P}{\Delta \forall / \forall_0}$$

But the general formula for static pressure with variable density is

$$\Delta P = \int_0^H \rho g \, dh$$

The density variable can be written as a function of bulk modulus:

$$\rho = \frac{m}{V} = \frac{m}{V_0 + \Delta V} = \frac{m}{V_0 - \frac{\Delta P}{K} V_0} = \frac{m}{V_0} \left(\frac{1}{1 - \frac{\Delta P}{K}} \right) = \rho_0 \left(\frac{1}{1 - \frac{\Delta P}{K}} \right)$$

Which is a function of ΔP , the change in pressure:

$$\Delta P = \int_0^H \rho g \, dh = \int_0^H \frac{\rho_0}{1 - \frac{\Delta P}{K}} g \, dh$$

Here we see ΔP appearing on both sides of the equation. We could work a bit harder to find a direct solution, or we can solve this *implicit* solution iteratively. Treating ΔP as a constant:

$$\Delta P = \int_0^H \rho g \, dh = \int_0^H \frac{\rho_0}{1 - \frac{\Delta P}{K}} g \, dh = \frac{\rho_0 g H}{1 - \frac{\Delta P}{K}}$$

ΔP still appears on both sides, but we can guess a value on the RHS and use that to calculate and recalculate the LHS.

A first guess of ΔP is as per part a) , $\Delta P = \rho g H = 1002(9.8)2000 = 19.64 \times 10^6 \text{ Pa}$

A second guess would then be $\Delta P = \rho g H / (1 - \Delta P / K) = 1002(9.8)2000 / (1 - 19.64 \times 10^6 / 2.05 \times 10^9) = 19.83 \times 10^6$

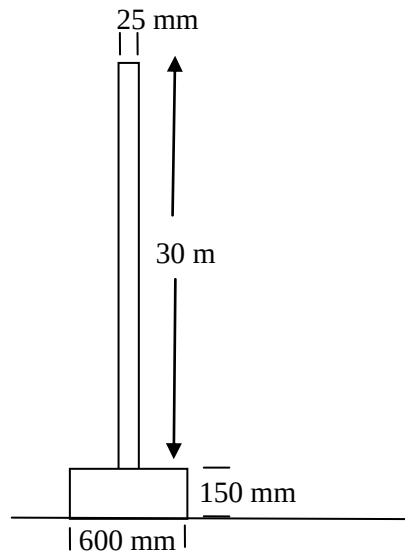
A third guess would be $\Delta P = \rho g H / (1 - \Delta P / K) = 1002(9.8)2000 / (1 - 19.83 \times 10^6 / 2.05 \times 10^9) = 19.83 \times 10^6$

By the third iteration we find the solution equals the guess, which completes the solution. This may seem almost miraculous if you have never used the iterative technique before, but it is well worth understanding as a quick way of solving formulae where the unknown cannot be isolated.

The solution can be interpreted physically as an increase on the static pressure (constant density) solution, because of the extra mass crammed into 2000m depth due to increasing density as a result of compression. This shows (by an apparently circular definition) how the iterative (or circular) solution makes physical sense.

Someone explains to you that the absolute pressure in a liquid of constant density doubles when the depth is doubled. Should you agree? Why or why not?

A vertical pipe 30 m long and 25 mm diameter has its lower end open and flush with the inner surface of the cover of a box which is 0.15 m high, square in plan view, each side being 0.6 m long. The bottom of the box is horizontal. If the box and pipe are completely filled with water calculate (a) the hydrostatic force exerted on the bottom of the box, (b) the force exerted on the floor upon which the box rests excluding the weight of the box and the pipe:



a) The hydrostatic pressure at the bottom of the box is $P = \rho g H = 9800(30.15) = 295.5 \times 10^3 \text{ Pa}$

The base of the box is square, of area $A = 0.6^2 = 0.36 \text{ m}^2$

The hydrostatic force on the bottom is thus $F = PA = 295.5 \times 10^3(0.36) = 106.4 \times 10^3 \text{ N}$

b) This is not the same as weight force of the structure:

The total volume of the system is $V = \Sigma Ah = 0.36(0.15) + \pi(0.025^2/4)(30) = 68.73 \times 10^{-3} \text{ m}^3$

The mass of the system is $M = \rho V = 1000(68.73 \times 10^{-3}) = 68.73 \text{ kg}$

Which creates a weight force of $F = Mg = 9.8(68.73) = 673.5 \text{ N}$

This is an illustration of the somewhat paradoxical nature of hydrostatic pressure. What's the difference between hydrostatic force and weight force? The difference is in which parts of the free body support which parts of the load. Hydrostatic force is a force in containment, that is it acts on the walls of the container. The box will need to be designed such that the *welded edges* are capable of supporting (containing) a hydrostatic force of 106.4 kN, while the *floor* that the box stands on must be capable of supporting (carrying) a much smaller weight force of 673 N. The tank *contains* the hydrostatic pressure force, while the floor *supports* the weight.

You might have to think about that for a while...

A manometer connected to a pipe in which a fluid is flowing indicates a negative gauge pressure of 50 mm of mercury (density 13600 kg/m^3). What is the absolute pressure in the pipe if the atmospheric pressure is 1 bar?

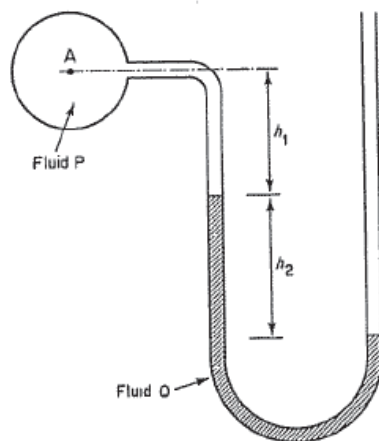
The mercury manometer indicates the *gauge* pressure in proportion to its density:

$$P = \rho g H = 13600(9.8)0.050 = 6664 \text{ Pa (gauge static pressure)}$$

The absolute pressure is the gauge pressure PLUS the atmospheric pressure. 1 Bar is approximately 1 standard atmosphere, or 100,000 Pa (100 kPa). Thus the absolute pressure is

$$P_{\text{abs}} = P_{\text{gauge}} + P_{\text{atm}} = 6664 + 100,000 = 106,664 \text{ Pa}$$

Fluid P is air and fluid Q is oil of specific gravity 0.8. If $h_1 = 300 \text{ mm}$ and $h_2 = 50 \text{ mm}$ (a) find the gauge pressure at A. (b) Determine the absolute pressure at A if the barometer reads 740 mm of mercury



Air has such a low density compared to any liquid that we can treat it equal to zero in situations like this. The difference in pressure between the air chamber and the external atmosphere is thus indicated by the height h_2 of fluid Q. But this is in a state of suction, since the water level has been drawn up towards the air chamber, so the gauge pressure is negative:

$$P_{\text{gauge}} = -\rho g h = -800(9.8)0.05 = -392 \text{ Pa}$$

With the barometer (density $13,500 \text{ kg/m}^3$) at 740 mm we can determine atmospheric pressure:

$$P_{\text{atm}} = \rho g H = 13570(9.8)0.74 = 98.4 \text{ kPa}$$

The absolute pressure in the bulb is thus $98.4 \times 10^3 - 392 = 98.01 \text{ kPa}$

The pressure head in a static gas main at a point 120 m above sea level is equivalent to 180 mm of water. Assuming that the densities of air and gas remain constant at 1.202 kg/m^3 and 0.561 kg/m^3 respectively, what will be the pressure head in mm of water at sea level?

This rather confusingly stated problem is nevertheless typical of run-of-the-mill engineering questions. Pressure head is by definition a gauge pressure, so we know that the pressure difference between inside and outside of the pipeline would raise water by 180 mm, as in the deflection of a water manometer fitted to the pipeline. This is a gauge pressure of

$$P_{\text{gauge}} = \rho g H = 9800(0.18) = 1764 \text{ Pa}$$

120 m lower, the pressure head in the pipeline is increased by the static pressure of that height of gas:

$$\Delta P_p = \rho g H = 0.561(9.8)120 = 659.7 \text{ Pa}$$

The pressure of atmospheric air outside the pipeline is increased in the same way, except the density is higher outside so the pressure increase is greater:

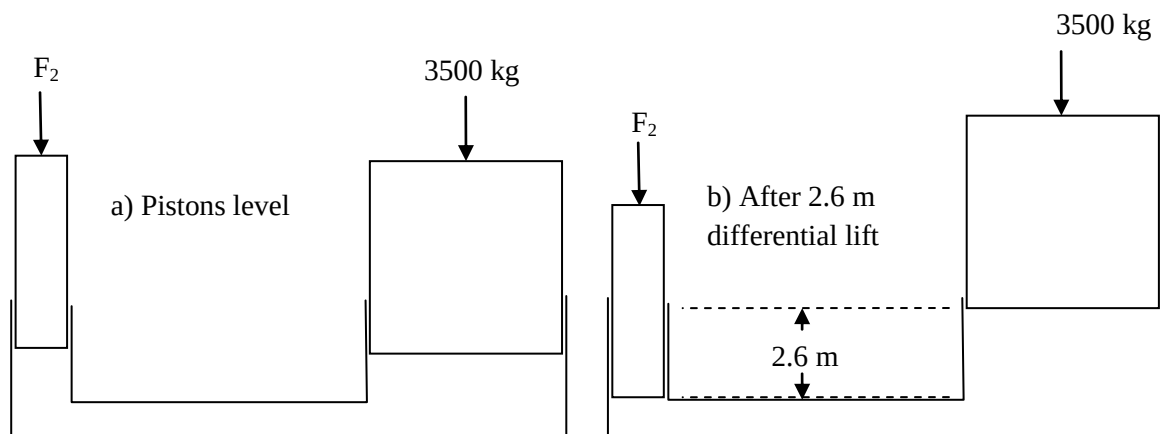
$$\Delta P_a = \rho g H = 1.202(9.8)120 = 1414 \text{ Pa}$$

The external pressure thus increases in excess of the internal pressure by $1414 - 659.7 = 754.3 \text{ Pa}$. The internal pressure decreases in comparison to the external pressure, so the gauge pressure is reduced by that amount:

$$P_{\text{sealevel}} = P_{\text{gauge}} - (\Delta P_a - \Delta P_p) = 1764 - 754.3 = 1010 \text{ Pa}$$

As mm of water, that is $H = P/\rho g = 1010/9800 = 103 \text{ mm}$

A hydraulic jack has a diameter ratio between the two pistons of 8 to 1. The diameter of the smaller piston is 600 mm. The larger piston is required to support a load of 3500 kg. The jack's working fluid is hydraulic oil of specific gravity 0.8. Calculate the force required on the smaller piston to provide the required lift (a) when the two pistons are level, and (b) when the smaller piston is 2.6 m below the larger piston.



a) With the pistons level we only need to balance the forces due to the different piston areas. The larger diameter is $0.6 \times 8 = 4.8$ m. The corresponding piston areas are:

$$A_1 = \pi 4.8^2 / 4 = 18.1 \text{ m}^2$$

$$A_2 = \pi 0.6^2 / 4 = 0.2827 \text{ m}^2$$

The working fluid has a constant pressure throughout:

$$P = F_1/A_1 = F_2/A_2$$

$$F_1/A_1 = mg/A = 3500(9.8)/18.1 = 1896 \text{ Pa}$$

$$F_2/A_2 = 1896 = F_2/0.2827$$

$$F_2 = 1896(0.2827) = 535.7 \text{ N}$$

This corresponds to a mass of $535.7/9.8 = 54.66$ kg

A 55 kg weight will level balance a 3500 kg weight in this device.

b) The 2.6 m lift adds an additional amount to the force required to lift the piston, equal to the weight of working fluid in the larger piston over that height. This can again be worked out by a balance of pressures:

$$P = 3500(9.8)/18.1 + 800(9.8)2.6 = 22.28 \times 10^3 \text{ Pa}$$

$$P = F_2/A_2 : 22.28 \times 10^3 = F_2/0.2827$$

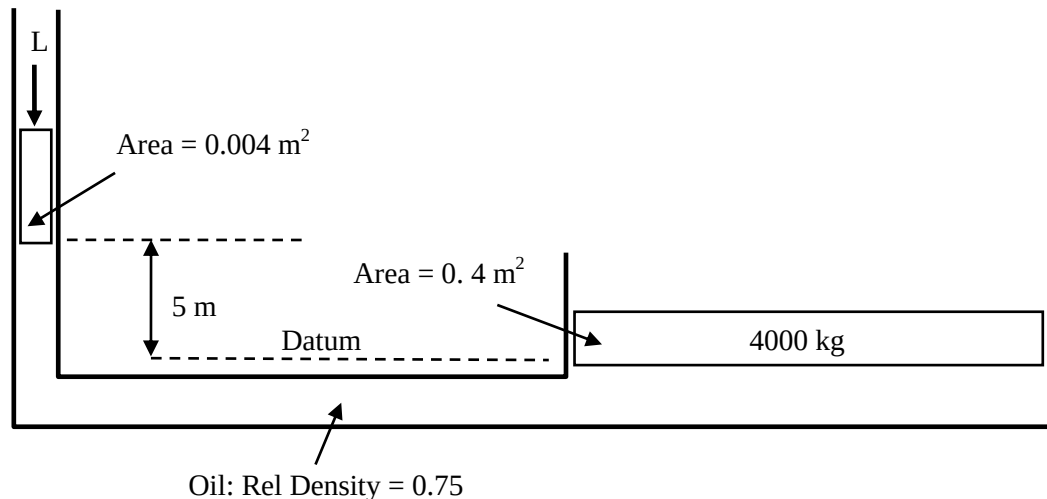
$$F_2 = 22.28 \times 10^3 \times 0.2827 = 6299 \text{ N}$$

This corresponds to a mass of $6299/9.8 = 642.7$ kg

It can be seen that a hydraulic jack works only over a limited lift range, as the weight of the working fluid comes to dominate the force balance.

The diagram shows a hydraulic jack that uses hydrostatic pressure to amplify force. You can assume that the seals (not shown in the diagram) are effective at containing the hydraulic oil inside the jack.

- What is the pressure at the base of the 4000 kg load?
- To what height above datum will the lift piston rise with no lift load (ie $L = 0$)?
- What force L must then be applied to achieve the lift configuration shown?
- By how much does the 4000 kg load rise after application of L ? Should you correct your calculation in c) to allow for this?



- $$P = F/A$$

$$= mg/A$$

$$= 4000(9.8)/0.4 = \underline{98 \text{ kPa}}$$
- The pressure at the datum is constant throughout the fluid, so a hydrostatic force due to an oil column of height H at L will balance the load.

$$P = \rho g H = 98 \times 10^3$$

$$\text{So } H = 98 \times 10^3 / (9.8(750)) = \underline{13.133 \text{ m}}$$

- If an additional load L is added, this creates an additional pressure that reduces the jack column height to 5 m as shown.

$$P_L = \text{pressure due to } L = L/A = L/0.004$$

$$P_H = \text{pressure due to hydrostatic load} = \rho g H = 9.8(750) 5 = 36.8 \text{ kPa}$$

$$\text{But } P_L + P_H = 98 \times 10^3$$

$$\text{Or } P_L = 98 \times 10^3 - 36.8 \times 10^3 = 61.2 \times 10^3$$

So $P_L = L/0.004 = 61.2 \times 10^3$

Thus $L = 0.004(61.2 \times 10^3) = \underline{245 \text{ N}}$

This is only 25 kg mass!

- d) The load L falls from 13.33 to 5 m, a fall of 8.33 m. This displaces a volume of oil:

Volume = $A.H = 0.004(8.33) = 0.033 \text{ m}^3$

The 4000 kg mass rises by the same displaced volume, over a distance H :

$V = 0.033 = A.H = 0.4H$

Thus $H = 0.033/0.4 = \underline{0.083 \text{ m}}$

This is a fairly small height, but it will change the position of the load wrt datum and so will introduce additional hydrostatic forces. The relative proportion of the additional forces is as 8.3 cm in 5 m, or 1.6%. This is an acceptably small error, so no recalculation is needed.

A researcher injects 1 ml of distilled water into a length of 3 mm internal diameter PVC tubing, and then seals the other end by folding over and clamping so that a 1 m length of tubing is occupied by water and trapped air. The tube is then pinned to a wall so as to hang vertically with the 1 ml of water at the bottom end (see Figure 1).

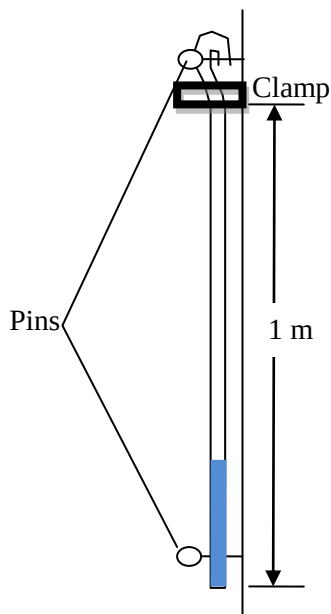


Figure 1: A 1 ml H_2O slug suspended in a 3 mm diameter tube (open to atmosphere at bottom end)

a) Why doesn't the water drain out when the tube is hung vertically? What's holding it up? Is this a stable configuration? Is there any mechanism by which the water can escape from the tube?

b) How deep is the 1 ml water column? What gauge pressure does this create in the trapped air inside the tube? Why is the gauge P negative?

c) The initial temperature and atmospheric pressure are 25°C and 90,000 Pa. If the temperature falls by 10°C , to what height will the base of the water column rise? Take the bottom of the tube as your zero height or *datum*. (Hint: use the ideal gas law $PV/T = \text{constant}$ and use initial values to evaluate the constant. Use absolute T and P)

d) If the temperature stays the same (as initial T) but the atmospheric pressure rises by 1000 Pa, to what height will the water column rise? (Hint: solve for the internal pressure as an absolute rather than gauge pressure, and relate this to the ideal gas law with the same constant)

a) The water is held up by suction in the air column. Constrained by the PVC, the water can't escape except by stretching the air (which can expand, as a gas) and it doesn't have enough weight to do that.

This is stable so long as the system isn't subjected to pressure shocks – a windy day with an open window is enough to shock the water into leaking out spontaneously. The shocks overcome the small potential energy barrier due to surface tension, and the lower energy state of a leak occurs.

The main problem is that the water evaporates at about 5 mm/day.

b) $\underline{h} = V/A = 1 \times 10^{-6} / (\pi D^2/4) = 1 \times 10^{-6} / (\pi(3 \times 10^{-3})^2/4) = \underline{0.14 \text{ m}}$

Pressure contributions from one end of the system (external) through static pressure to the other (internal) we have:

Atmospheric Pressure – ρgh = Internal Pressure

The static pressure is negative as the direction from A is upwards ie to less depth.

So – ρgh = Internal Pressure - Atmospheric Pressure = Gauge Pressure

The gauge pressure is as defined in lecture 2, the difference between the absolute internal and atmospheric (external) pressures.

So Gauge Pressure = – ρgh = – 9800(0.14) = – 1372 Pa = – 1.4kPa

This is negative since the air column is in suction. Considering the very small mass of water involved, this is a significant pressure (due to the small diameter of the tube).

For more accurate work, use $g = 9.807 \text{ m/s}^2$ and $\rho_w = 1002 \text{ kg/m}^3$, so $\rho g = 9827 \text{ N/m}^3$

c) The ideal gas law $PV/T = \text{constant}$, can be solved for the constant:

$k = PV/T$

The Initial volume of the air is the length of 1 m less the water column:

$V = (\pi D^2/4) (1 - 0.14) = (\pi(3 \times 10^{-3})^2/4) 0.86 = 6.1 \times 10^{-6} = 6.1 \text{ ml OK}$

The initial absolute internal pressure is

$P = 90000 - 9800(0.14) = 88628 \text{ Pa}$

So $k = 88628(6.1 \times 10^{-6})/298 = 0.0018$

The new volume at 15°C is found by

$V = kT/P = 0.0018(288)/88628 = 5.9 \times 10^{-6} = 5.9 \text{ ml OK}$

Which is a length contraction of $(5.9 - 6.1) \times 10^{-6} / (\pi D^2/4) = -0.028 \text{ m} = \text{a 2.8 cm rise in the water column:}$

$\Delta h_T = 2.8 \text{ cm}$

d) The absolute internal pressure at 91000 Pa atmospheric is

$$P = 91000 - 9800(0.14) = 89628 \text{ Pa}$$

The new volume is found by

$$V = kT/P = 0.0018(298)/89628 = 6.0 \times 10^{-6} = 6.0 \text{ ml OK}$$

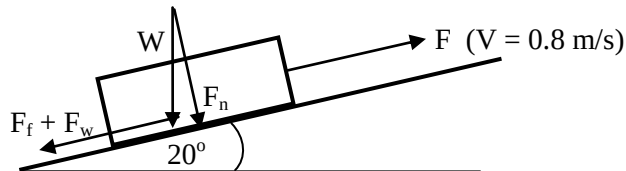
Which is a length contraction of $(6.0 - 6.1) \times 10^{-6} / (\pi D^2/4) = -0.014 \text{ m} = \text{a } 1.4 \text{ cm rise in the water column.}$

$$\underline{\Delta h_p = 1.4 \text{ cm}}$$

Some solutions came in lower at around 0.9 cm. I checked, and it seems to be a sensitivity to rounding in this problem. So the true answer is a range from 0.9 to 1.4 cm.

Viscosity

A block 50 cm long, 30 cm wide and 20 cm high weighing $W = 150 \text{ N}$ is to be moved at a constant velocity of 0.8 m/s on an inclined plane surface with an active friction coefficient of $k = 0.27$ where $F_f = k F_n$ and F_w is the plane-parallel component of the weight force. Determine the steady force F required to move the block. If a 0.4 mm thick oil film with viscosity 0.012 Pas is applied between the block and the surface, determine the percentage reduction in the required horizontal force. Repeat for a heavier oil 1 mm thick, viscosity 0.05 Pas .



In the unlubricated situation the velocity of the block is independent of the friction force, which is a function only of the normal force and friction factor. We require an initially high force to overcome static friction and accelerate the mass to the desired velocity, but once equilibrium is reached the propelling force is only required to balance weight forces and active friction:

The normal force F_n is given by:

$$F_n = W \cos \theta = 150 \cos(20) = 141 \text{ N}$$

The friction force is the normal force multiplied by a friction factor of $K = 0.27$:

$$F_f = k F_n = 0.27(141) = 38.06 \text{ N}$$

The friction force is aligned to the component of weight force parallel to the plane, F_w :

$$F_w = W \sin \theta = 150 \sin(20) = 51.3 \text{ N}$$

The total force (unlubricated) is thus

$$F = F_f + F_w = 38.07 + 51.3 = 89.37 \text{ N}$$

With a lubricant of $0.012 \text{ Pa}\cdot\text{s}$ and 0.4 mm thick, the friction force is replaced by a shear force equal to the viscous shear stress multiplied by the area of the block:

$$F_t = \tau A$$

$$\text{where } \tau = \mu \frac{dV}{dy}$$

here $\mu = 0.012$, $dV = 0.8 \text{ m/s}$ and $dy = 0.4 \text{ mm}$: (the velocity gradient dV/dy can be approximated by the velocity and the depth of flow)

$$\text{thus } \tau = 0.012(0.8)/0.4 \times 10^{-3} = 24 \text{ Pa}$$

$$\text{and } F_t = \tau A = 24(0.5)0.3 = 3.6 \text{ N}$$

The lubricated motive force is thus $F = F_t + F_w = 3.6 + 51.3 = 54.9 \text{ N}$

The reduction can be expressed as a percentage:

$$\text{Reduction\%} = (54.9 - 89.4) / 89.4 \times 100\% = -38.6\%$$

A similar calculation pertains to the heavier oil:

$$\tau = 0.05(0.8) / 1 \times 10^{-3} = 40 \text{ Pa}$$

and $F_t = \tau A = 40(0.5)0.3 = 6.0 \text{ N}$

thus $F = F_t + F_w = 6.0 + 51.3 = 57.3 \text{ N}$

so $\text{Reduction\%} = (57.3 / 89.4 - 1) \times 100 = -35.9\%$

The space between two large flat and parallel walls 25 mm apart is filled with a liquid of absolute viscosity 0.7 Pa s. Within this space a thin flat plate, 250 mm x 250 mm is towed at a velocity of 150 mm/s at a distance of 6 mm from one wall, the plate and its movement being parallel to the walls. Assuming linear variations of velocity between the plate and the walls, determine the force exerted by the liquid on the plate.

The force is generated by viscous shear as determined by Newton's Law:

$$\tau = \mu \, dV/dy \quad \text{viscous shear}$$

$$F = \tau A \quad \text{shear force}$$

The plate has two separate layers of shear which must be calculated separately: since the two sides have the same area the calculations can be summed directly.

$$V = 0.15 \text{ m/s}$$

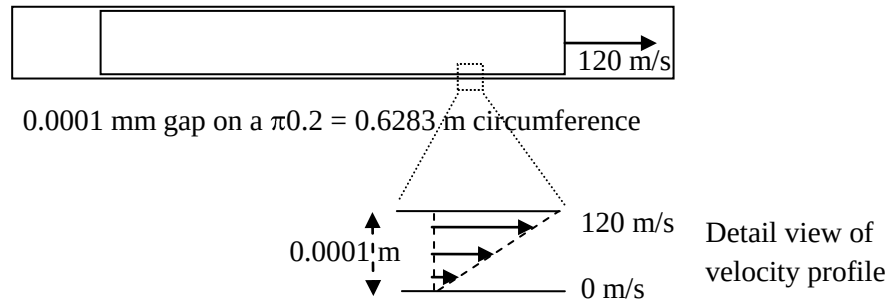
$$y_1 = 0.006 \text{ m} \quad dV/dy_1 = 0.15/0.006 = 25 \quad \tau_1 = 0.7(25) = 17.5$$

$$y_2 = 25 - 6 \text{ mm} = 0.019 \text{ m} \quad dV/dy_2 = 0.15/0.019 = 7.9 \quad \tau_2 = 0.7(7.9) = 5.53$$

$$F = A(\tau_1 + \tau_2) = 0.25^2(17.5 + 5.53) = \underline{1.44 \text{ N}}$$

A hydraulic ram 200 mm in diameter and 1.2 m long moves within a cylindrical casing 200.2 mm in diameter. The piston is lubricated with oil of relative density 0.85 and kinematic viscosity $400 \times 10^{-6} \text{ m}^2/\text{s}$. What is the viscous force resisting the motion when the piston moves at 120 m/s?

It helps to draw a diagram:



The ram is subject to viscous friction according to Newton's law:

$$\tau = \mu \frac{dV}{dy}$$

τ is a shear stress or lateral pressure with the units of pressure, Pa, which can be multiplied by an area to produce a force. The relevant area is the surface area of the piston in lubricated contact with the casing:

$$F = \tau A$$

$$A = L\pi D = 1.2(0.6283) = 0.754 \text{ m}^2$$

The velocity gradient dV/dy is assumed to be linear due to the no-slip condition on both the ram and the casing surfaces. The relative velocity on both surfaces is zero, so the velocity at the casing is zero, and on the ram it is 120 m/s, with a linear variation between. The velocity gradient can thus be estimated by the ram velocity and the gap distance:

$$dV/dy = 120/0.0001 = 1.2 \times 10^6$$

The viscosity can be derived from the kinematic viscosity and the density, as

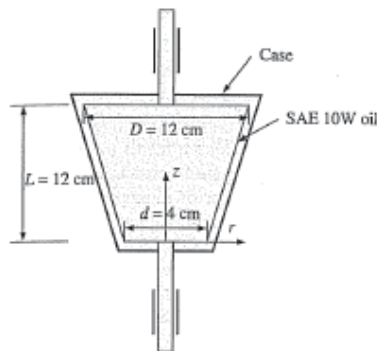
$$\nu = \mu/\rho$$

$$\mu = \rho \nu = 400 \times 10^{-6} \times 850 = 0.34 \text{ Pa.s}$$

$$\text{So } F = \tau A = \mu A dV/dy = 0.34(0.754)1.2 \times 10^6 = 307.6 \times 10^3 \text{ N}$$

This very high motive force is due to the high velocity of the piston.

A plug-shaped body is rotating at a constant angular speed of 200 rad/s in a casing filled with SAE 10W oil at 20 °C ($\mu = 0.1 \text{ Pa}\cdot\text{s}$). If the clearance between the casing and the body is 1.2 mm determine the power required to maintain this motion.



This problem requires integration of shear stress across elements of area, where both the shearing area and the shearing velocity vary with the diameter of the plug. The trick is to distinguish macroscopic variables that can be defined in real terms at any point, such as the shearing velocity at any diameter, from *infinitesimal* variables, such as the element of area at any diameter.

Usually in these problems you need to decide or define which variable is the primary infinitesimal, that is which variable can be reduced to a dx for integration. In this case the length of the cone $S = 0.12 \text{ m}$ can be subdivided into elements of length dS which can be multiplied by a varying circumference to get varying elements of area:

$$dA = 2\pi r dS \text{ where } r \text{ varies from } 0.06 \text{ to } 0.02 \text{ m over the length } S. (dA = \text{an element of area})$$

Then an infinitesimal Torque or rotational force can be defined by a shear stress acting over the infinitesimal area:

$$dT = \tau dA$$

The total torque can be found by integrating the infinitesimal torque dT , and the total power can be estimated by multiplying the total torque by the angular velocity:

$$\text{Power} = \omega T = \omega \int dT \quad (\text{see first year physics: circular motion})$$

That's a quick overview, the general idea. To proceed, we need to break dT down into components that can be evaluated at any value of S , the length variable:

$$dT = \tau dA = \tau 2\pi r dS \text{ as before.}$$

So far so good. Now τ is the friction shear stress that creates the torque:

$\tau = \mu V/t$ where μ is the viscosity, V is the linear velocity and t is the gap clearance. μ and t are constants, but V varies with radius:

$$V = r\omega \text{ thus } \tau = \mu\omega/t$$

$$\text{So } dT = \tau dA = \mu\omega/t 2\pi r dS = 2\pi \mu\omega/t r^2 dS$$

The last step separates the constant terms μ , ω and t from the variable terms r and S . To integrate this we just need to show how r varies as a function of S :

ie r varies from 0.06 m at $S = 0$ to 0.02 m at $S = 0.12$

This is a straight line: $r = 0.06 + kS = 0.02$ @ $S = 0.12$, thus $k = (0.02-0.06)/0.12 = -0.333$

ie $r = 0.06 - S/3$

and $r^2 = (0.06 - S/3)(0.06 - S/3) = 0.0036 - 0.04S + S^2/9$

so $dT = 2\pi \mu \omega / t (0.0036 - 0.04S + S^2/9) dS$

This can now be integrated by the polynomial rule (first year maths):

$$T = \int dT = 2\pi \mu \omega / t [0.0036 S - 0.04S^2/2 + S^3/27] \text{ evaluated at } S = 0 \text{ and } 0.12$$

Substituting all the known values ,

$$T = 2\pi 0.1(200)/0.0012 [0.0036(0.12) - 0.04(0.12)^2/2 + 0.12^3/27] = \underline{21.78 \text{ N.m}}$$

And the power is

$$\text{Power} = \omega T = 200 (21.78) = \underline{4.356 \text{ kW}}$$

NOTE: Linear integration problems such as this can often be approximated by averages, but you have to be careful to identify which variables can be averaged, and realise that co-variability can create large inaccuracies. In this case the average radius is 4 cm, so the average velocity is

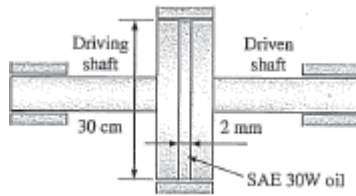
$$r\omega = 0.04 \times 200 = 8 \text{ m/s (average - approximate)}$$

Likewise the area can be calculated by averages: $A = 2\pi rS = 2\pi(0.04)0.12 = 0.0302 \text{ m}^2$

So the torque can be calculated as $T = \tau A = \mu V/t A = 0.1(8)/0.0012 (0.0302) = \underline{20.1 \text{ N.m}}$ (cf 21.78)

This is an 8% inaccuracy. *Be careful*: 8% is about as accurate as this type of short cut gets.

A clutch system is used to transmit torque through a 2 mm thick SAE 30W oil film with $\mu = 0.38 \text{ Pa}\cdot\text{s}$. between two 30 cm diameter disks. When the drive shaft is moving at 1450 RPM the driven shaft is observed to rotate at 1398 RPM. Assuming a linear velocity profile, determine the transmitted torque. Repeat for 5 mm and 10 mm clearances



A Newtonian fluid flows down a shallow inclined plane. The upper surface is in contact with the air, which offers no measurable resistance to the flow. Using Newton's law of viscosity, identify a suitable estimate of the rate of shear deformation (rotation) at the surface. Is the flow velocity a linear function of flow depth in this case?

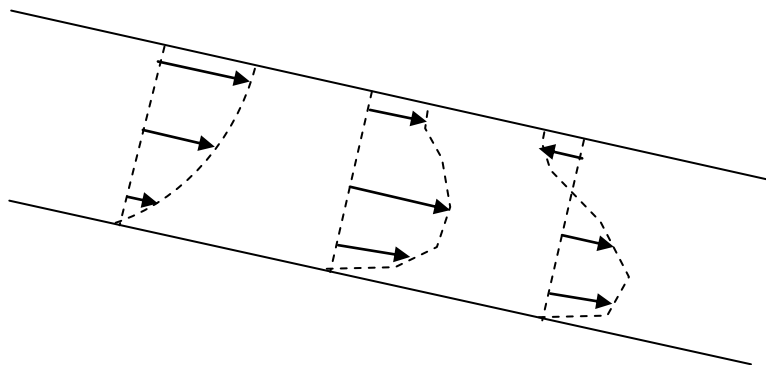
Since there is no resistance at the surface, there can be no shear stress at that point and so, using Newton's law of viscosity we must find the velocity gradient there is zero:

$$\tau = 0 \text{ thus } dV/dy = 0 \text{ at the surface}$$

The sheet of fluid (let's call it water) flows at a steady velocity due to viscous shear creating a resistance to gravity forces. The shear is likely to be greatest at the base of the flow, thus $dV/dy \gg 0$ at the *invert* or bed of the flow:

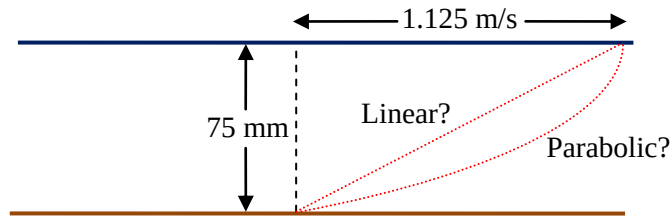
$$\tau = \mu \frac{dV}{dy} \gg 0 \text{ thus } dV/dy \gg 0 \text{ at the bed.}$$

We can sketch a range of velocity profiles that match these conditions:



While the velocity profile may do some very interesting things, including adverse (backwards) flow in some locations plus three-dimensional elements such as vortices and eddies, the only linear profile that can arise is the still-water case, where $V=0$ throughout the depth. As water cannot be stationary on an inclined slope as shown, *a linear velocity profile is not possible for flow with a free surface.*

A fluid with viscosity 0.048 Pa.s and relative density 0.913 flows 75 mm deep with an open surface as shown. The velocity varies from zero at the floor to a maximum value of 1.125 m/s at the surface. The velocity profile may be linear or parabolic as shown in the diagram. Calculate the velocity gradient and shear stress at the wall under both assumptions.



Under an assumed linear velocity profile the velocity gradient everywhere is just the maximum velocity divided by the maximum depth:

$$\frac{dV}{dy} = \frac{V_{max}}{Y} = 1.125 / 0.075 = 15 /s$$

The shear stress is then

$$\tau = \mu \frac{dV}{dY} = 0.048(15) = 0.72 \text{ Pa}$$

But we have already seen that this is impossible!

An assumed parabolic profile with minimum assumptions is

$$V = V_{max} \left(1 - \left(\frac{Y-y}{Y} \right)^2 \right)$$

This can be differentiated:

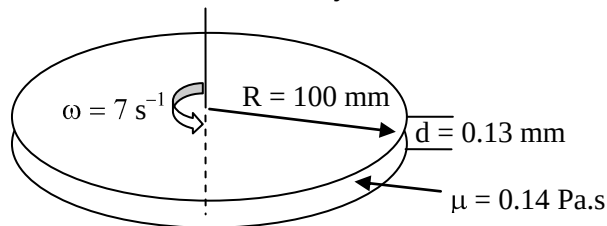
$$\frac{dV}{dy} = -2V_{max} \left(\frac{Y-y}{Y} \right) \left(\frac{-1}{Y} \right) = 2V_{max} \left(\frac{Y-y}{Y^2} \right)$$

At $y = 0$ this equates to $\frac{dV}{dy} = \frac{2V_{max}}{Y} = 2(1.125)/0.075 = 30 /s$

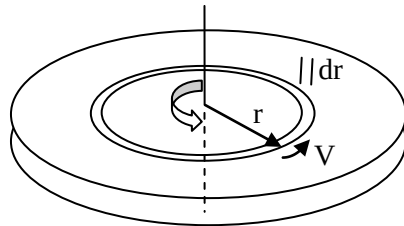
The shear stress is $\tau = 0.048(30) = 1.44 \text{ Pa}$

The maximum shear stress in parabolic flow (actually a realistic flow in some circumstances) is thus double what you would expect for a linear velocity profile. So apart from being impossible, the linear profile is inaccurate even as a simplifying assumption by at least a factor of 2x!

A uniform film of oil 0.13 mm thick separates two discs, each of 200 mm diameter, mounted co-axially. Ignoring edge effects, calculate the torque necessary to rotate one disc relative to the other at a speed of 7 rev/s if the oil has a viscosity of 0.14 Pa s.



Worked solution: Using a circular integration method to add up all elements of shear force:



Length of circular element, $l = 2\pi r$

Linear velocity of circular element, $V = \omega r$ (after converting ω to radians s^{-1})

Element of Area, $dA = 2\pi r dr$

Shear stress, $\tau = \mu \frac{dV}{dy} = \mu \frac{V}{d}$ (V the linear velocity of each circular element, d the disk spacing)

$$= \mu \frac{\omega r}{d}$$

Element of shear force $dF = \tau dA = \mu \frac{\omega r}{d} \cdot 2\pi r dr = 2\pi \mu \frac{\omega}{d} \cdot r^2 dr$

This force element creates a torque element dT around the axis of rotation:

$$dT = dF \cdot r = \mu \frac{\omega r}{d} \cdot 2\pi r dr \cdot r = 2\pi \mu \frac{\omega}{d} \cdot r^3 dr$$

Integrating,

$$T = 2\pi \mu \frac{\omega}{d} \int_0^R r^3 dr = 2\pi \mu \frac{\omega}{d} \left[\frac{r^4}{4} \right]_0^R = \frac{1}{2} \pi \mu \frac{\omega}{d} R^4$$

$\mu = 0.14 \text{ Pa.s}$

$\omega = 7 \text{ s}^{-1} = 7 \times 2\pi \text{ rad s}^{-1} = 43.98 \text{ rad s}^{-1}$

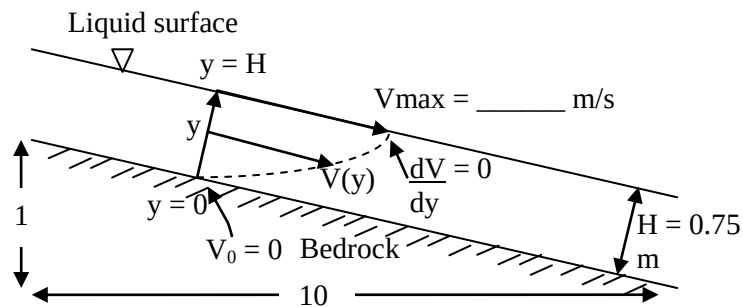
$d = 0.13 \times 10^{-3} \text{ m}$

$R = 0.1 \text{ m}$

Thus
$$T = \frac{1}{2} \pi (0.14) \frac{43.98}{0.13 \times 10^{-3}} (0.1)^4 = \underline{7.4 \text{ N.m}}$$

From a distance of over a kilometre, a team of researchers observe a fresh lava flow on a hillside known to be a smooth inclined plane with a 1:10 slope. By noting the height of the lava against a landmark boulder they estimate the depth of the lava flow to be about 0.75 m. They time the flow over a distance of 50 m between two other known rocks, noting that it takes 26 seconds for a particular cinder (a tree stump probably) to travel that distance. They also note that the lava is extremely hot and fluid, visibly rolling its surface as it flows.

The team leader makes this sketch in a notebook:
(Figure 2: Cross section through theoretical lava flow)



- Why the curved velocity profile? What type of basic curved mathematical function could fit that?
- What is $V_{\max}$? Assuming your profile curve is right, derive an expression for the velocity as a function of elevation y and depth H . Parameterise with $V_{\max}$ and 1 other constant.
- Assuming your theory is correct, what is the shear rate dV/dy of the flow at its base? Why is $dV/dy = 0$ at the surface?
- Using a lava density of 2700 kg/m^3 , what is the shear stress on the bedrock? Is that the same as the shear stress in the lava at that level ie $y = 0$?
- Given the shear stress in d) and the basal shear rate in c), estimate the viscosity of the lava at this temperature.

a) The velocity profile has zero slope at the air surface, as air presents no appreciable friction to any liquid over short distances and speeds. The velocity must be zero at bedrock, due to Newton's original no-slip observation. Also at $y = 0$ the frictional shear stress is highest, and hence the velocity profile slope or shear rate is steepest. This means the shear stress increases increasingly with depth, like a **parabola.**

b) $V_{\max} = 50 \text{ m} / 26 \text{ s} = 1.9 \text{ m/s}$

$V = V_{\max} - k(H - y)^2 = 1.9 - k(H - y)^2$ would make sense as a parabola, since $V = V_{\max}$ at $y = H$, so long as

$$K = 1.9/(H)^2 = 1.9/(0.75)^2 = 3.38 \text{ by solving where } y = 0$$

Thus $V = 1.9 - 3.38(H - y)^2 \text{ m/s}$ as a function of y and H Other formulations are possible, and accepted in marking.

c) $dV/dy = -3.38 \times 2(H - y) \times -1 = 6.76(H - y)$ (Polynomial rule)

but $dV/dy = 6.76(H - y) = 0$ @ $y = H$ ie at the surface the friction slope goes to zero as per a)

and $dV/dy = 6.76H = 6.76(0.75) = 5.07 /s$ is the shear rate at $y = 0$ ie at bedrock – maximum shear rate

That's 5.07 m/s per m of depth, which is a fairly large shear rate (imagine trying to step onto a moving walkway that goes from zero to 18 km/hour in 1 step).

d) Shear Stress τ is the pressure of weight along the slope (rather than into the face) of the surface: Shear stress under a mass in either moving or stationary equilibrium is just the static pressure multiplied by the slope (or sin of the angle to be precise, but let's use the slope for order of magnitude accuracy):

$$\tau = \rho g H S = 2700(9.8)0.75(0.1) = 1,985 \text{ Pa}$$

The shear stress of the bedrock and the state of shear in the lava sheet are in equilibrium (no acceleration), therefore they are equal. The lava is eroding the bedrock. This is the same process as water erosion only much harsher.

e) Using Newton's Law of Viscosity,

$$\tau = \mu dV/dy$$

We have τ and dV/dy at one spot, $y = 0$, so we can solve the viscosity:

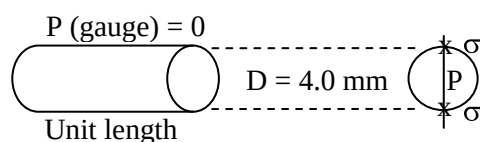
$$\mu = \tau/(dV/dy) = 1985/5.07 = 392 \text{ Pa.s}$$

This is quite low for lava. That stuff is hot!!

Surface Tension and Capillarity

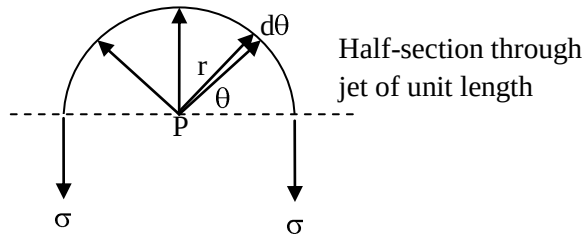
Estimate the height to which water at 20 °C will rise in a capillary tube of diameter 0.3 mm.

By how much does the pressure P in a cylindrical jet of water 4 mm in diameter exceed the pressure of the surrounding atmosphere if the surface tension of water is $\sigma = 0.073 \text{ N/m}$?



The surface tension is expressed as a circumferential force per unit length of the jet. Like a tightened ring, the circumferential force σ is balanced by an additional internal pressure P . As the diagram suggests, the circumferential force can be visualised by looking end-on to a 1m projection of the jet and picturing the surface tension forces as acting in the circumferential direction, and the pressure acting against it in the radial direction.

Again using a circular integration scheme, we can balance elements of pressure force on one side of the jet against the circumferential surface tension:



$$dF = PdA = P r d\theta$$

$$dF_{\text{vertical}} = dF \sin \theta = P r \sin \theta d\theta$$

Integrating,

$$dF_{\text{vertical}} = Pr \int_0^{\pi} \sin \theta .d\theta = Pr \left[-\cos \theta \right]_0^{\pi} = Pr \left[-(-1) - (-1) \right] = 2 Pr$$

This pressure force is balanced by a tension force acting on both sides of the half-cylinder:

That is $2Pr = 2\sigma$

$$\text{Or } P = \sigma/r = 0.073 / 2 \times 10^{-3} = \underline{36.5 \text{ Pa}}$$

A 0.2 mm air bubble forms inside a liquid. Determine the pressure difference between the inside and outside of the bubble if the surface tension of the liquid is 0.08 N/m.

A surface tension force derives from chemical forces on the longest linear dimension of a fluid surface, in this case the circumference of the bubble. This linear force is balanced by an induced pressure inside the bubble, which acts on the corresponding area in the plane of the circumference:

$$\text{Surface tension force: } F_{\sigma} = \sigma l = \sigma \pi D = 0.08 \pi 0.2 \times 10^{-3} = 50.27 \times 10^{-6} \text{ N}$$

$$\text{Pressure force: } F_p = PA = P \pi D^2/4 = P(31.42 \times 10^{-9})$$

The two forces must be in balance:

$$F_{\sigma} = 50.27 \times 10^{-6} = F_p = P(31.42 \times 10^{-9})$$

The pressure force is thus

$$P = 50.27 \times 10^{-6} / 31.42 \times 10^{-9} = 1600 \text{ Pa}$$

Smaller bubbles have correspondingly higher pressures. Very small bubbles can thus create a surprisingly high pressure environment.

A 0.1 mm droplet of water is suspended in a rising air flow. Determine the gauge pressure inside the droplet if the surface tension of water at this temperature is 0.07 N/m

A surface tension force derives from chemical forces on the longest linear dimension of a fluid surface, in this case the circumference of the droplet. This linear force is balanced by an induced pressure inside the droplet, which acts on the corresponding area in the plane of the circumference:

Surface tension force: $F_\sigma = \sigma l = \sigma \pi D = 0.07 \pi 0.1 \times 10^{-3} = 21.99 \times 10^{-6} \text{ N}$

Pressure force: $F_p = P A = P \pi D^2 / 4 = P (7.85 \times 10^{-9})$

The two forces must be in balance:

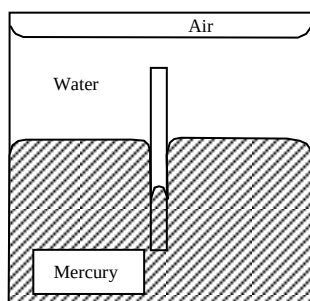
$$F_\sigma = 21.99 \times 10^{-6} = F_p = P (7.85 \times 10^{-9})$$

The pressure force is thus

$$P = 21.99 \times 10^{-6} / 7.85 \times 10^{-9} = 2801 \text{ Pa}$$

Smaller droplets have correspondingly higher pressures. Very small droplets can thus create a surprisingly high pressure environment.

What is the approximate capillary rise of mercury, with relative density 13.56, surface tension $\sigma = 0.377 \text{ N/m}$ and contact angle 140° , in contact with water in a clean glass tube 6 mm in diameter? (Note: as the mercury rises it displaces water)



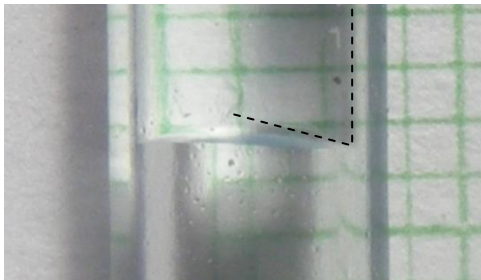
A 1.2 mm diameter tube is inserted into an unknown liquid of density 960 kg/m^3 . It is observed that the liquid rises 5 mm inside the tube and forms a contact angle of 15° to the vertical. Determine the surface tension in N/m of the liquid.

The surface tension of a liquid is to be measured using a liquid film suspended on a U-shaped wire frame with an 8 cm long moveable side. If the force needed to move the wire is 0.024 N, determine the surface tension of the liquid (hint: consider both sides of the thin film as separate surfaces)

Estimate the maximum diameter of a solid steel ball, density 7800 kg/m^3 , that can float on water at 20°C due to surface tension effects. Compare this to the maximum diameter of a floating aluminium ball, density 2700 kg/m^3 .

A researcher takes some close-up photos of the upper and lower meniscus in a water column (Figure 3):

1)



2)

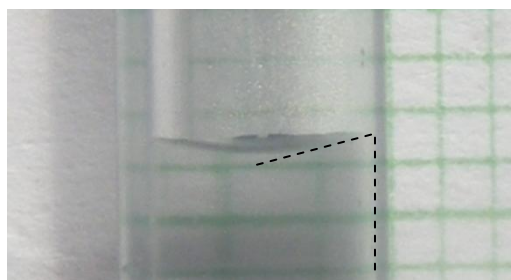


Figure 3: Meniscus, distilled water in 3 mm PVC tube; 1) bottom of water column 2) top of column. Grid squares 1 mm. Note droplets $\sim 50\mu\text{m}$ suspended against gravity.

- Is surface tension a significant factor in the forces that hold the water column up against gravity? (60 words min/max)
- After a quick Google scholar search I find 2 factors of surface energy, the surface tension of water $\sigma_{\text{air-water}} = 0.073 \text{ N/m}$ and the solid surface tension of PVC $\sigma_{\text{air-pvc}} = 0.039 \text{ N/m}$. Given that the contact angle is measured at 74° (sketched – check it with a protractor), calculate the surface energy of the water-PVC contact surface ($\sigma_{\text{water-pvc}}$) in J/m^2 or N/m .
- Seeking to solve the instrument's basic problem as identified in 1a), the researcher uses SAE-30 monograde engine oil instead of water to create a liquid column inside the PVC:

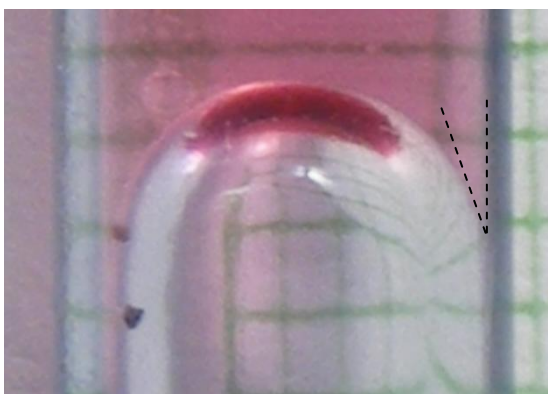


Figure 4: Meniscus, SAE-30 in 3 mm PVC tube; 1) bottom of column 2) top of column. Note leakage out of bottom left of column

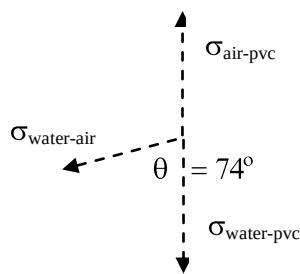
- continued: The top meniscus has a contact angle of about 20° (sketched). Looking up the oil surface tension $\sigma_{\text{air-oil}} = 0.036 \text{ N/m}$, and knowing the PVC surface tension from above, what is the surface energy of the oil-PVC interface ($\sigma_{\text{oil-pvc}}$) in J/m^2 or N/m ?

d) The researcher notes with some frustration that the oil, unlike the water, leaks out by slowly oozing down the inside of the PVC. Explain this in terms of a minimum surface energy state. Does it agree with your rough calculation in c)? Why is the bottom meniscus different?

a) The contact angle is large, around 74° . The surface tension force is as the cosine of that angle, and projects only about 0.28 of the surface tension force. Regardless of the angle, the top and bottom meniscus are identical, so they project equal opposite forces which do not contribute to the force balance. Also the tension force is as the circumference of the tubing, while the weight force is as the area – so for capillaries much larger than 1 mm the water tension capillary lift is negligible.

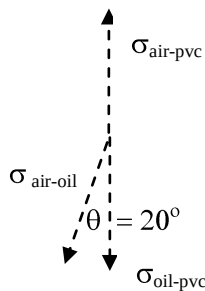
b) The tensile forces are in balance as per Young's (contact angle) equation:

$$\sigma_{\text{air-pvc}} = \sigma_{\text{water-pvc}} + \sigma_{\text{air-water}} \cos(\theta)$$



$$\text{Thus } \underline{\sigma_{\text{water-pvc}} = \sigma_{\text{air-pvc}} - \sigma_{\text{air-water}} \cos(\theta)}$$

$$= 0.039 - 0.073 \cos(74) = \underline{\underline{0.019 \text{ N/m}}}$$



c) Similarly for the oil

$$\sigma_{\text{air-pvc}} = \sigma_{\text{oil-pvc}} + \sigma_{\text{air-oil}} \cos(\theta)$$

$$\text{Thus } \underline{\sigma_{\text{oil-pvc}} = \sigma_{\text{air-pvc}} - \sigma_{\text{air-oil}} \cos(\theta)}$$

$$= 0.039 - 0.036 \cos(20) = \underline{\underline{0.005 \text{ N/m}}}$$

The oil-PVC surface Energy is $0.005 \text{ J/m}^2 = \text{N/m}$

d) The leakage is evidence that a move that increases oil-PVC contact by “wetting” (or oiling rather) the PVC surface, is a move to a more stable state, which by definition is a lower energy level. Both the Oil-PVC and the Oil-Air energies are lower than the corresponding Water-PVC system, which seems to agree.

Also the smaller contact angle indicates a higher affinity for the liquid to spread over the surface.

It seems the oil leaks because it is easier, ie it takes less energy, to leak than not to. Whereas in the case of water on PVC it's the other way around.

(The bottom meniscus actually shows the leakage, and is not at any contact angle but an asymptote of $\theta = 0$)