
Water Waves

The mathematical description of waves in water varies from quite simple and easily accessible to horrendously complex and almost totally impenetrable. In practice, the actual behaviour of water waves is uncertain and conforms to the theoretical description only in the broadest statistical sense.

This supplement is designed to take you from the simple to the complex, starting with basic ideas already established in open channel and real fluid theory, and progressing through to a range of theoretical and practical topics.

Very few derivations are given. Even when the theoretical results are reasonably straightforward the methods of deriving them are always mathematically advanced. The approach taken is to understand the theory through examples and illustrations or by exploring behaviour over a suitable range.

It is assumed, that if you are tackling this topic in the teaching semester you are doing so out of pure interest in a fascinating topic. Don't tackle it too early, as you will need to know most of the material in the unit to even partly understand it!

Gravity Waves in Shallow Water: River Waves

In open channel hydraulics we have seen, without derivation (see last page of this supplement), that the celerity of a surface wave is given by:

$$c = \sqrt{gy} \quad \text{Celerity of a river wave (m/s)}$$

where y is the mean depth of fluid. This can be understood as satisfying a balance between pressure and momentum forces, such that the effects of density are balanced, ie they cancel out. Thus as the depth of flow increases, the velocity of a wave in shallow water (or indeed any low-viscosity liquid) increases in proportion only to the square root of the depth.

Note that the celerity is independent of the wave height, the wavelength or any other feature of the wave. In fact (as we shall see) this simple description is an approximation that applies only to waves of height between about 1.5% and 10% of the flow depth. Smaller waves may be effected by surface tension forces (capillary waves), or in deep water (as in ocean waves) they become functions of the wavelength rather than the flow depth. Larger waves are subject to nonlinearities and are dependent on the height as well as the depth (see derivation on the last page).

River celerity is relative to the flow velocity of the fluid. As viewed by an observer on the bank, the apparent celerity of a wave depends on whether the wave is travelling upstream or downstream, and whether the flow is in the subcritical or supercritical flow regime:

$$c_{\text{apparent}} = \sqrt{gy} \pm V \quad \text{Apparent celerity allowing for flow velocity (m/s)}$$

So in subcritical flow a wave travelling upstream is retarded by the flow velocity, while a wave travelling downstream is advanced. In supercritical flow where $V > c$ a wave cannot travel upstream, and the entire structure of waves as stable moving influences is distorted by high levels of turbulence as well as the presence of standing waves arising from bed formations.

Significant river waves are generated by interactions between the current and obstructions or irregularities in the bed and the riverbank. The energy of a wave derives from the energy of the interaction that created it, and is transmitted as a form of power at the velocity of the wave front.

Note that a gravity wave is different from a momentum-continuity wave such as a surge or bore. These are a type of mobile hydraulic jump and can be derived from open channel theory.

Gravity Waves in Deep Water: Ocean Waves

With a little thought it should be apparent that in very deep water the celerity cannot continue to depend on depth. A wave is a surface phenomenon, creating pressure imbalances, momentum exchanges and friction reactions that effect only a shallow region near the surface. The motion of fluid particles in a wavefront diminishes rapidly with depth, such that beyond a certain depth any motion in the fluid is unaffected by the wave. Similarly the wave's kinetic energy must be unaffected by fluid reactions at that depth, or by the depth itself.

To see this, note that the surface of even the wildest ocean is approximately smooth compared to the 8000+ m depth. A 10 m wave is only 0.09% of the 11,030 m Marianas trench.

Gravity waves in relatively deep water are known as *ocean waves*, whose celerity is given by:

$$c = \sqrt{\frac{gL}{2\pi}} \quad \text{Celerity of an ocean wave (m/s)}$$

Where L is the wavelength (ie the distance from peak to peak). Again, a balance of pressure and momentum forces results in the cancellation of factors such as density and wave height, leaving only the wave length as a square root. It is perhaps remarkable that a factor of 2π also remains, since this is a mathematical rather than a physical parameter.

Ocean waves are generated by complex interactions between the water surface and winds or atmospheric pressure. Turbulent boundary layers in the wind are modified by the increasing roughness of the water surface, which in turn modifies the rate at which energy develops in the waves. Atmospheric pressure systems generate very low frequency waves of the order of tens to hundreds of kilometres in wavelength. Similar to tides, long waves can be significant variables in harbour design.

Waves transmit atmospheric energy over great distances, with the result that waves deriving from completely different wind and pressure systems will often combine and cross while travelling in different directions. Waves at any location are thus a mixture of multiple wave sources, having different energies and different parameters derived from energy. Even waves from a common source have a range of energies due to the complexity and variability of the interaction with atmosphere.

As a result, ocean waves are not constant in wavelength or wave height, they are complex mixtures or *spectra* of wave parameters, producing irregular patterns that can only be described statistically. In particular a mixture of wavelengths occurs, and since the celerity depends on wavelength there is no single celerity but a mixture of celerities, often moving in different directions.

This mixture of celerities means that ocean waves are *dispersive*: over time the faster, long waves will overtake the short waves and leave them behind. The complex mixture of wave energies disperses and sorts itself into a simpler pattern, for example waves from a storm far out at sea will arrive at shore in order of wavelength, longest first. This can be used to give an indication of just how far out at sea the storm was.

Ocean waves tend to travel in *wave groups* between nodes of relatively calm water. A common observation is that waves arriving at the shore go through a cycle of large and small waves (the largest wave is not necessarily the seventh). Wave groups travel at *half* the mean celerity of waves in the group, with the result that waves seem to form at the back of the group, grow and then diminish as they move through the group, and disappear again at the front of the group. The total wave energy is contained as a superposition of interacting waves, and is transmitted at the group celerity.

Gravity Waves at Intermediate Depths: Hyperbolic Waves

Clearly there must be a transition from a shallow “river” wave to a deep “ocean” wave. For example as an ocean wave approaches a shoreline the effect of the shallower depths will be to produce an interaction between the wave and the sea bed at depth y while weakening the dependence on L . At some intermediate depth the wave celerity must be a function of both mean depth *and* wavelength.

In general, or in intermediate situations where the shallow and deep approximations are not valid, the celerity is given by:

$$c = \sqrt{\frac{gL}{2\pi} \tanh\left(\frac{2\pi y}{L}\right)} \quad \text{Celerity of the general gravity wave (m/s)}$$

Where $\tanh$ (pronounced *than* as in *thought*) is the hyperbolic tangent:

$$\tanh(x) = \frac{e^{2x} + 1}{e^{2x} - 1}$$

Note that for small values of x , $\tanh(x) \sim x$: thus as the flow depth y becomes small relative to the wavelength L (shallow water):

$$c = \sqrt{\frac{gL}{2\pi} \tanh\left(\frac{2\pi y}{L}\right)} \longrightarrow \sqrt{\frac{gL}{2\pi} \left(\frac{2\pi y}{L}\right)} = \sqrt{gy}$$

Also note that for large values of x , $\tanh(x) \sim 1$: thus as the flow depth y becomes large relative to the wavelength L (deep water):

$$c = \sqrt{\frac{gL}{2\pi} \tanh\left(\frac{2\pi y}{L}\right)} \longrightarrow \sqrt{\frac{gL}{2\pi}}$$

This “explains” the difference between ocean waves and river waves: they are simply two extremes of the unified gravity wave.

“Deep” and “shallow” are relative to the depth and wavelength. The ripples on a swimming pool are far shorter than the depth, so they behave as ocean waves and travel much more slowly than a splash wave say from a bomb-dive, which has a wavelength approximately the size of the body producing it.

The Effect of Surface Tension: Capillary Waves:

Very small waves (of the order of a couple of centimetres or less) are effected by surface tension. In fact all waves are effected by surface tension, but the effect is only noticeable for very small waves.

To see this, we can now introduce the full linear wave model (Airy, 1857):

$$c = \sqrt{\left(\frac{gL}{2\pi} + \frac{2\pi\sigma}{\rho L}\right) \tanh\left(\frac{2\pi y}{L}\right)} \quad \text{Celerity of a linear wave (m/s)}$$

Where σ is the surface tension (about 0.075 N/m for water) and ρ is the density. The surface tension term contains a dimensionless number governing surface tension and body forces, the Weber number:

$$We = \frac{\rho L}{\sigma}$$

$$\text{Thus } c = \sqrt{\left(\frac{gL}{2\pi} + \frac{2\pi}{We}\right) \tanh\left(\frac{2\pi y}{L}\right)}$$

Since We is a large number for all but very small lengths, surface tension has no effect on typical river or ocean waves other than in the very first stages of wind generation.

Also the relation y/L is usually large and so $\tanh(2\pi y/L) \rightarrow 1$ as for ocean waves:

$$\text{Thus } c = \sqrt{\left(\frac{gL}{2\pi} + \frac{2\pi\sigma}{\rho L}\right)} \quad \text{Celerity of a capillary wave (m/s)}$$

For wavelengths less than about 1.7 cm the surface tension force becomes dominant and celerity increases with *decreasing* wavelength. This implies that there must be a *minimum celerity* at the transition from gravity wave to capillary wave. To see this, differentiate by L and find the root:

$$\begin{aligned} \frac{dc}{dL} &= \frac{1}{2} \left(\frac{gL}{2\pi} + \frac{2\pi\sigma}{\rho L} \right)^{-1/2} \left(\frac{g}{2\pi} - \frac{2\pi\sigma}{\rho L^2} \right) \\ &= 0 \quad @ \quad \frac{g}{2\pi} = \frac{2\pi\sigma}{\rho L^2} \end{aligned}$$

$$\text{ie } L = \sqrt{\frac{4\pi^2\sigma}{\rho g}} = 2\pi \sqrt{\frac{\sigma}{\rho g}}$$

For water, this has a value of:

$$L = 2\pi\sqrt{(0.075/9800)} = 0.017 \text{ m}$$

Thus the minimum celerity of the most general type of water wave occurs at a wavelength of 1.7 cm:

$$c_{min} = 0.23 \text{ m/s } @ L = 0.017 \text{ m}$$

Surface tension is effected by surfactants (which reduce σ) and oils (increase σ). A technique for calming water in ocean rescues is to pour oil on the water, producing waves of longer period.

Wave Displacements

An artificially generated linear wave of homogenous wavelength and frequency has a water surface profile that conforms closely to the sinusoidal wave law familiar from first-year physics:

$$\eta(x, t) = A \cos\left(\frac{2\pi x}{L} - \frac{2\pi t}{T}\right) \quad \text{Surface displacement as a function of distance, time (m, s)}$$

Where A is the wave amplitude or maximum vertical displacement from mean depth, T is the wave period or time between successive crests, and x, t are positions in space and time along the wave series. The symbol η is eta, equivalent to a Greek h .

The terms inside the cosine brackets can be reduced to:

$$\frac{2\pi}{L} = k \quad k \text{ the wave number}$$

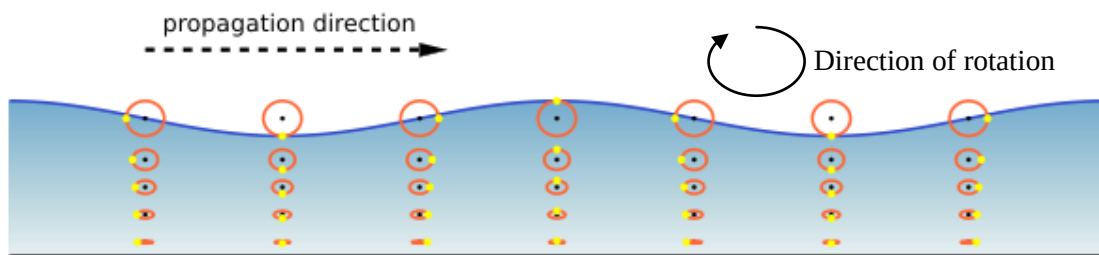
$$\frac{2\pi}{T} = 2\pi f = \omega \quad \omega \text{ the angular frequency for } f \text{ the wave frequency}$$

$$\text{Or } \eta(x, t) = A \cos(kx - \omega t)$$

The wavelength L and period T are related to the celerity c as $c = \omega/k = L/T$. Note that there is as yet no relationship between the amplitude and the wavelength.

The wave function conveys *energy*, but there is no net translation or transport of water particles. Observation of wave motion reveals an oscillatory motion (try floating behind the surf line), with forward motion on the crest of the wave and an equal backward motion in the trough.

The Airy or linear wave theory confirms this and shows that the motion is elliptical, tending to circular in deep water and near the surface, with the ellipse flattening to a horizontal back-forth motion at the bed in shallow water:



Elliptical motion of fluid particles (source: Wikipedia)

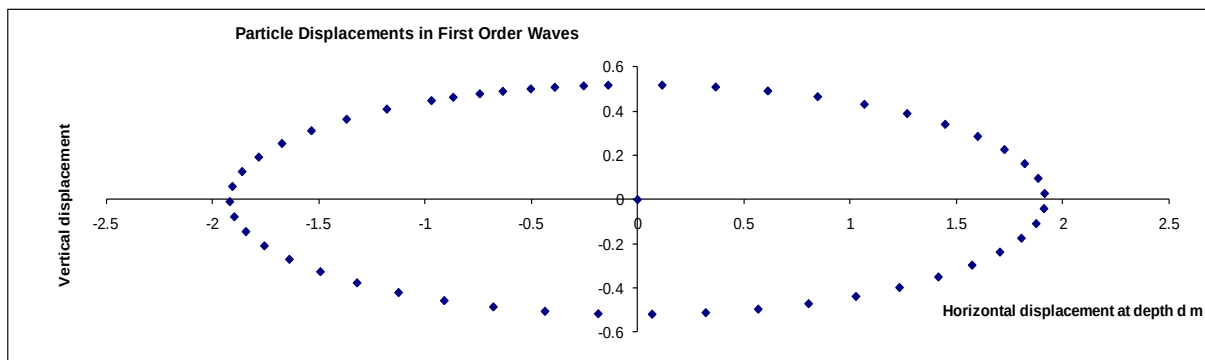
The elliptical motion can be calculated by two linked equations as follows. In the figure above each ellipse is centred on a point with local coordinates $(x, z) = (0, 0)$ where x and z are horizontal and vertical displacements from that particular centre. Considering a centre at mean depth d under a wave of amplitude A , the displacement of fluid particles around that depth is given by:

$$x = \frac{A \cosh\left(\frac{2\pi(y-d)}{L}\right)}{\sinh\left(\frac{2\pi y}{L}\right)} \sin\left(\frac{2\pi t}{T}\right)$$

$$z = \frac{A \sinh\left(\frac{2\pi(y-d)}{L}\right)}{\sinh\left(\frac{2\pi y}{L}\right)} \sin\left(\frac{2\pi t}{T}\right)$$

(source: Chadwick and Morfett, 1993)

Where $\sinh$ and $\cosh$ (pron. *shine* and *co-shine*) are the hyperbolic sine and cosine functions and t is time in seconds.



Particle displacements at depth $d = 15$ m and mean depth $y = 20$ m under a wave of length $L = 113$ m and amplitude $A = 2.5$ m. At $d = 20$ m (ie at the sea bed), this curve flattens to a horizontal back-and-forth motion with amplitude of 1.8 m. This creates a considerable shear stress on the bed.

Wave Celerity and Period

We have $c = L/T$. Putting this into the hyperbolic wave celerity equation we get:

$$c = \frac{L}{T} = \sqrt{\frac{gL}{2\pi} \tanh\left(\frac{2\pi y}{L}\right)} = \sqrt{\frac{gcT}{2\pi} \tanh\left(\frac{2\pi y}{L}\right)}$$

Noting that $\frac{c}{\sqrt{c}} = \sqrt{c}$ we get $\sqrt{c} = \sqrt{\frac{gT}{2\pi} \tanh\left(\frac{2\pi y}{L}\right)}$ or

$$c = \frac{gT}{2\pi} \tanh\left(\frac{2\pi y}{L}\right) = \frac{g}{\omega} \tanh(ky) \quad \text{Wave celerity as a function of length, depth and period (m/s)}$$

In deep water, as before, the $\tanh$ term goes to the value of 1:

$$c_o = \frac{gT}{2\pi} = \frac{g}{\omega} \quad \text{Deep water wave celerity as a function of period}$$

The c_o subscript is added to indicate that ocean waves start out at this celerity, and then decelerate as they enter shallow water, ie

$$c = c_o \tanh\left(\frac{2\pi y}{L}\right)$$

Energy conservation requires that the period T is constant even as wavelength and celerity vary with depth. We are left with:

$$T = L/c = \text{constant}$$

$$\text{So} \quad \frac{c}{L} = \frac{gT}{2\pi L} \tanh\left(\frac{2\pi y}{L}\right) = \frac{g}{2\pi c} \tanh\left(\frac{2\pi y}{L}\right) = \text{constant}$$

A usefully constant relationship between celerity, depth and wavelength.

Wave Energy

The kinetic energy of a wave is the sum of all horizontal particle motions at depths below the surface, while the potential energy is the sum of all vertical mass displacements around the mean depth. The familiar rules of $E_k = mV^2/2$ and $E_p = mgh$ are applied as integrals over many particle elements.

Due to equipartition of energy the potential and kinetic energies of a wave turn out to be equal:

$$E_k = E_p = \frac{\rho g H^2 L}{16} \quad \text{Wave energy components per width of wave crest (J/m)}$$

Where H is the wave height, equal to twice the amplitude, $H = 2A$. This is the basic relationship between wavelength and amplitude. The total energy is:

$$E_t = E_k + E_p = \frac{\rho g H^2 L}{8} \quad \text{Total wave energy per width of wave crest (J/m)}$$

This is expressed as energy per unit width. It can be more usefully expressed as energy per unit area by dividing by the length:

$$E_t = \frac{\rho g H^2}{8} \quad \text{Wave surface energy (J/m}^2\text{)}$$

This total surface energy is carried not at the wave celerity but at the *group celerity*. We noted earlier that ocean waves travel in groups, whose celerity is half that of an individual wave. In general, the group celerity is given by:

$$c_g = \frac{c}{2} \left(1 + \frac{\frac{4\pi y}{L}}{\sinh\left(\frac{4\pi y}{L}\right)} \right) \quad \text{Wave group celerity or energy speed (m/s)}$$

$$c_g = \frac{c}{2} \quad \text{Deep water wave group celerity (m/s)}$$

$$c_g = c \quad \text{Shallow water wave group celerity (m/s)}$$

The *power* of a wave group is the rate at which energy is transmitted, given by:

$$\text{Power} = E_t c_g \quad \text{Wave power (Watts/m}^2\text{)}$$

Example: Deep water waves of average height 2 m and length 80 m enter a broad estuary of uniform depth 4m. What wavelength and height will the waves take in the shallow water?

In deep water the waves travel at a speed of:

$$c = \sqrt{\frac{gL}{2\pi}} = \sqrt{\frac{9.8(80)}{2\pi}} = 11.2 \text{ m/s}$$

The wave period is constant at $T = L/c = 80/11.2 = 7.1 \text{ s}$

The wave surface energy is:

$$E_t = \frac{\rho g H^2}{8} = \frac{9800(2)^2}{8} = 4.9 \text{ kJ/m}^2$$

The wave group celerity is just half the celerity in deep water: $c_g = 11.2/2 = 5.6 \text{ m/s}$

The wave power is thus:

$$\text{Power} = E_t c_g = 4.9(5.6) = 27.4 \text{ kW/m}^2$$

To extract that total power would require a mechanism operating on both kinetic and potential energy principles. In practice, real world mechanisms manage only half that, minus inefficiencies.

In the shallow water of the estuary the wave celerity = group celerity is

$$c_g = c = \sqrt{gy} = \sqrt{9.8(4)} = 6.3 \text{ m/s}$$

The estuary wavelength is derived from the constant period: $L = cT = 6.3(7.1) = \underline{44.7 \text{ m}}$.

Assuming no losses, the wave power is still $27.4 \text{ kW/m}^2 = c\rho g H^2/8$

$$\text{Thus } H = \sqrt{\frac{8P}{\rho g c}} = \sqrt{\frac{27.4 \times 10^3 \times 8}{9800(6.3)}} = \underline{1.88 \text{ m}}$$

Once the estuary wavelength L has been estimated the celerity can be calculated with the full hyperbolic formula for a more accurate estimate of H and L . In this case after two iterations the solution is $H = 1.94 \text{ m}$ and $L = 42.5 \text{ m}$

Wave Refraction, Shoaling and Breaking: The Surf Zone

Waves approaching a shoreline are distorted by interactions with the sea bed. Assuming an approximately constant sea floor slope S_o , the depth and celerity decrease as per:

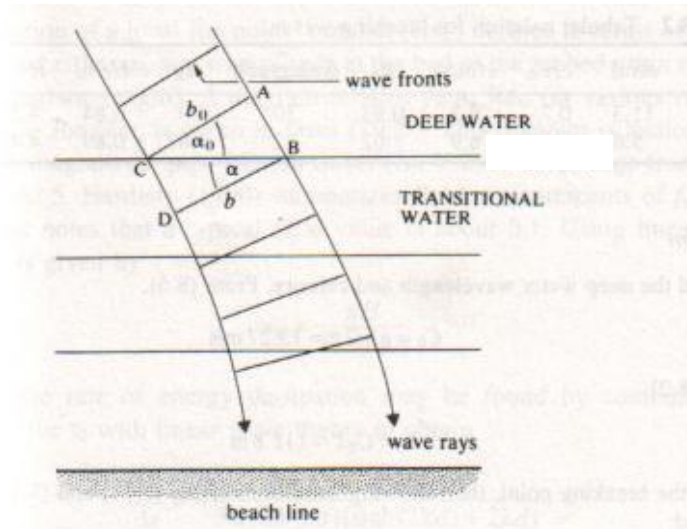
$$c = \frac{gT}{2\pi} \tanh\left(\frac{2\pi y}{L}\right) \rightarrow \sqrt{gy} = \sqrt{gS_o D} \text{ where } D \text{ is the distance from shore.}$$

Since the period T is constant (energy conservation), the wavelength L must also decrease:

$$L = cT = T\sqrt{gy} = L_o \tanh\left(\frac{2\pi y}{T\sqrt{gy}}\right) = L_o \tanh\left(\frac{2\pi\sqrt{y}}{T\sqrt{g}}\right)$$

The length and celerity decrease in the same proportion. The wave decelerates and compresses such that the period and wave energy are the same.

If waves approach the shore at an angle, the geometric compression of the wave occurs at different rates at different points along each wavefront: the waves *refract*, turning towards the beach:



Refraction of wavefronts approaching a beach (source: Chadwick and Morfett, 1993)

Without derivation, the change (ratio) in angle is given by:

$$\frac{\sin \alpha}{\sin \alpha_o} = \tanh\left(\frac{2\pi y}{L}\right) = \tanh\left(\frac{2\pi y}{T\sqrt{gy}}\right) = \tanh\left(\frac{2\pi\sqrt{y}}{T\sqrt{g}}\right)$$

Since the wave power is constant (less minor friction losses), the wave height adjusts to accommodate changes in celerity and length:

$$H = H_o \left[\frac{\cos(\alpha_o)}{\cos(\alpha)} \right]^{\frac{1}{2}} \left[\frac{\frac{c_o}{2}}{\frac{c}{2} \left[1 + \frac{4\pi d / L}{\sinh(4\pi d / L)} \right]} \right]^{\frac{1}{2}}$$

The wave height tends to decrease somewhat in the approach, and then increases again towards the breaking point near the shoreline.

The shear stress on the sea bed increases dramatically under an approaching wave due to the horizontal motion of fluid particles near the bed. This can create an erosive force which is quite complex and difficult to manage. The shear force is theoretically given by:

$$\tau_b = 0.5 f_w \rho u_m^2 \quad \text{Bed shear stress (Pa)}$$

Where f_w is a friction factor, $f_w \sim 0.1$ in practice, and u_m is the maximum orbital velocity of the oscillatory motion:

$$u_m = \frac{\pi H}{T \sinh\left(\frac{2\pi y}{L}\right)} \quad \text{Maximum orbital velocity m/s}$$

Waves break for a number of reasons, summarised by the following conditions:

- When the *steepness ratio* $H/L < 0.17$ (ocean waves)
- When the *breaking depth ratio* $\gamma = H/y < 0.78$ (beach waves)
- When bed friction slows the base of a wave relative to its crest: the crest gets ahead of the base, and it has to fall over.

The breaking depth ratio $\gamma = H/y$ can vary from 0.4 to 1.2 depending on beach slope and wave type. A dimensionless number the *Iribarren number* composed of the sea bed slope and steepness ratio can be used to predict the breaking depth ratio:

$$\gamma = \xi_o^{0.17} + 0.08 \quad \text{Breaking depth ratio (m/m)}$$

Where $\xi_o = \frac{S_o}{\sqrt{\frac{H}{L}}}$ is the *Iribarren number* for deep water wave height and length.

Wave momentum causes a *wave setup* η_u to occur, whereby the mean height at the shoreline is higher than mean sea level. There is also a *set down* η_d , where the mean sea level slumps at the surf line:

$$\eta_d = -\frac{1}{8} \frac{2\pi H_b^2 / L}{\sinh(4\pi y / L)}$$

$$\eta_u = \frac{3}{8} \gamma^2 y_b + \eta_d$$

Wave momentum at an angle to the beach also generates a *longshore current*:

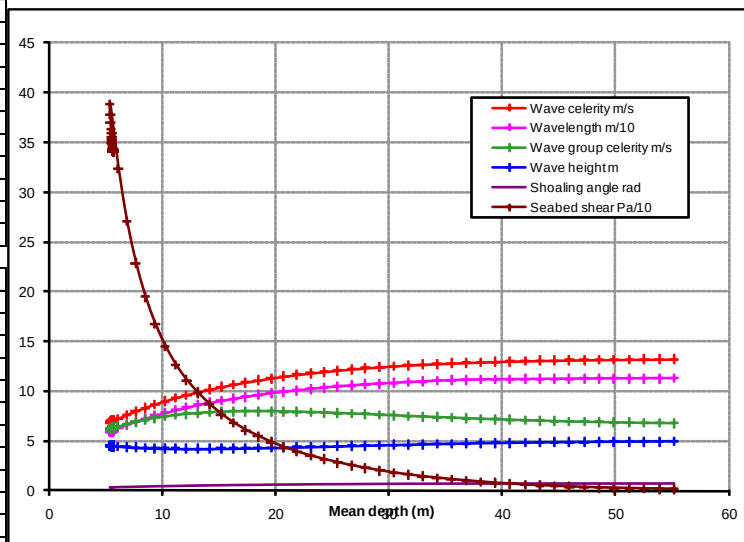
$$\bar{u}_l = 2.7 \frac{\gamma}{2} \sqrt{g y_b} \sin(\alpha_b) \cos(\alpha_b) \quad \text{Mean longshore current (m/s)}$$

Where the subscript b indicates values at the break line, ie in the surf.

It makes sense to perform these linked calculations in a spreadsheet, simulate them as a function of time, and plot the results:

Deep water wavelength	Lo	m	113.00
Deep water wave height	Ho	m	5
Angle wavefront to beach	α_0	rad	0.79
Shoaling rate	l/κ	m/m	10.00
Deep water wave celerity	co	m/s	13.28
Deep water wave group celerity	cg	m/s	6.64
Deep water frequency	ω_0	/s	0.74
Deep water period	T	s	8.51
Reset	r	r	
Time	T	s	366397
Interval	Δt	s	1
Wavelength Solution Tolerance	ΔL	m	2.00

Shoaling mean water depth	d	m	5.71
Shoaling wavelength	L	m	60.28
Shoaling angle	α	rad	0.39
Shoaling wave celerity	c	m/s	7.08
Shoaling wave group celerity	cg	m/s	6.37
Shoaling wave height	H	m	4.46
Seabed shear stress	τ_b	Pa	341
Test mean water depth	d	m	5.71
Scaled wavelength	Ls	m/10	6.03
Wave set down at breakline	η_d	m	-0.04
Wave setup at shoreline	η_u	m	1.27
Breaking near-bed orbital velocity	u_m	m/s	2.92
Longshore current	v_l	m/s	2.76



Spreadsheet model of wave shoaling and refraction. Each tick mark represents a 1-second simulation interval. Note the rapid increase in shear stress on the shear bed.

Wave Reflection

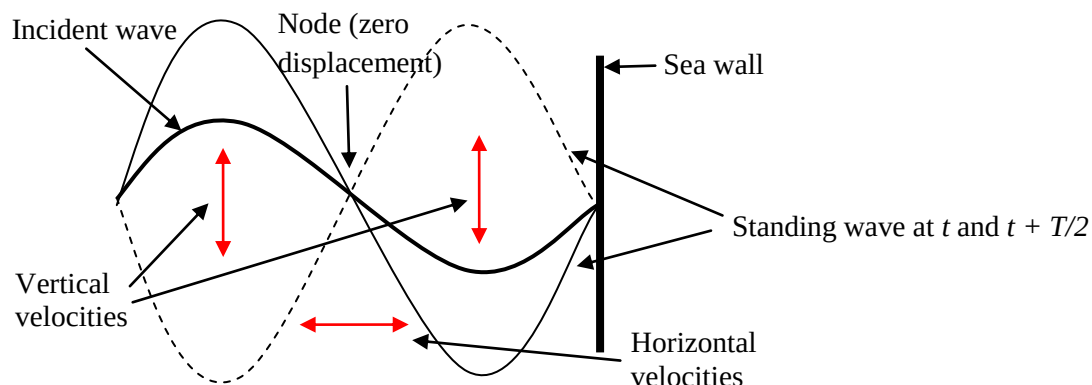
Water waves normally incident (ie at 90°) to a solid vertical boundary are reflected with the same phase but opposite direction. The phases add together constructively, doubling in amplitude. The result is a standing wave pattern that can be extremely difficult for small craft to navigate.

The standing wave pattern can be derived by adding together the incident and reflected waves:

$$\eta_i(x,t) = A \cos\left(\frac{2\pi x}{L} - \frac{2\pi t}{T}\right) \quad \text{Incident Wave}$$

$$\eta_r(x,t) = A \cos\left(\frac{2\pi x}{L} + \frac{2\pi t}{T}\right) \quad \text{Reflected Wave}$$

$$\eta_s(x,t) = \eta_i + \eta_r = 2A \cos\left(\frac{2\pi x}{L}\right) \cos\left(\frac{2\pi t}{T}\right) \quad \text{Standing Wave}$$



In addition to the large vertical velocities, the large volume of fluid in each crest is transferred across the adjacent node and back again in the period T , creating large, oscillating horizontal velocities. The mix of rapidly varying vertical and horizontal velocities create turbulence that is hazardous to small craft and very hazardous for a swimmer.

If the incident waves are steep then the collision with reflected waves can be a non-linear impact rather than a linear superposition. This creates a vertical plume of water known as a *clapotis*, or *clap* for short.

Waves incident at an angle α are reflected at the same angle, creating a diamond-shaped pattern of standing waves that propagate along the wall with an accompanying system of moving vortices. Very large erosive forces can develop in addition to the obvious danger to swimmers or small boats.

Not all barriers are impermeable vertical walls. Other types of barriers tend to reduce the reflected energy as described by a *reflection coefficient* $K_r = A_r/A_i$ (ratio of amplitudes):

Barrier type	K_r
Concrete sea wall	0.7 – 1.0
Rock Breakwater	0.2 – 0.7
Beach	0.05 – 0.2

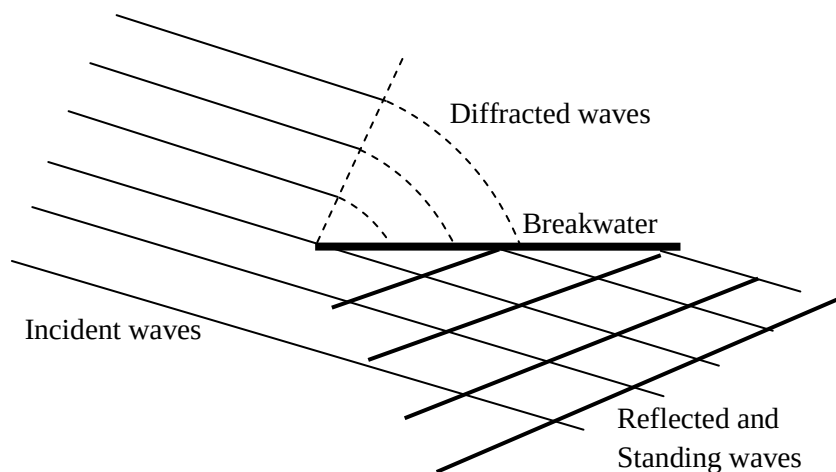
Or, in terms of the Iribarren Number,

$K_r = 0.125 \xi_p^{0.7}$ where ξ_p is the peak Iribarren number for approaching wavelengths:

$$\xi_p = \frac{S_o}{\sqrt{\max(H/L)}}$$

Wave Diffraction

Diffraction is the process whereby waves bend around corners by radiation of the wave energy. This occurs when wave fronts pass a barrier such as an island or the point of a breakwater at the mouth of a harbour. The linear wave fronts passing the obstruction peel off circular arcs that curl around the obstruction. The diffracted arcs are weaker as the energy is spread into the sheltered region behind the obstruction:



Seiche Waves

A *seiche* is a long period standing wave in a basin or harbour, which may be open to the sea or enclosed as in a lake. Harbour seiches are of particular concern since they may become amplified by waves entering through the harbour mouth. Long waves generated by atmospheric pressure systems are a particular problem for large harbours such as Rotterdam, which is around 20 km in length but only 20 m deep. Seiche barriers and forecasting are an active current field of research.

If the frequency of an entering wave train matches the natural oscillation frequency of a harbour the resulting seiche can begin to resonate, producing standing waves that increase in amplitude, causing damage to shipping or failure of the harbour structure. Even small vertical motions can produce large horizontal movements. If the horizontal oscillation matches the resonant frequency of moored ships, disaster inevitably results, breaking mooring lines or dashing ships against the moorings.

For a closed basin of characteristic dimensions (length or width) l and uniform depth y , we have

$$\eta_s(x,t) = A \cos\left(\frac{2\pi x}{L}\right) \cos\left(\frac{2\pi t}{T}\right) = A \cos(kx) \cos(\omega t) \quad (\text{standing wave})$$

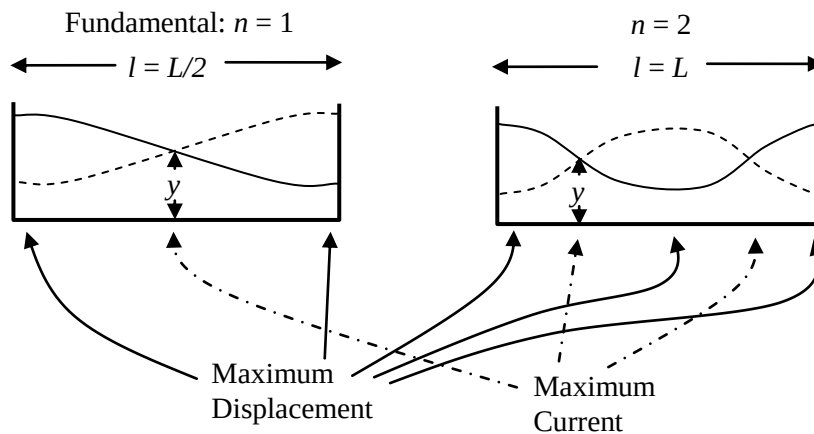
The resonant modes occur at a series of wave numbers:

$$k = \frac{2\pi}{L} = \frac{\pi}{l}, \frac{2\pi}{l}, \frac{3\pi}{l}, \dots, \frac{n\pi}{l}$$

These occur when the wavelength is a regular multiple of the characteristic length:

$$L = \frac{2l}{1}, \frac{2l}{2}, \frac{2l}{3}, \dots, \frac{2l}{n}$$

The first mode ($n = 1$) is known as the fundamental. Higher modes may occur in isolation or combination, resulting in complex standing waves. These are less likely than the fundamental.



Waves of some characteristic periods will resonate. To work out the periods, use the celerity:

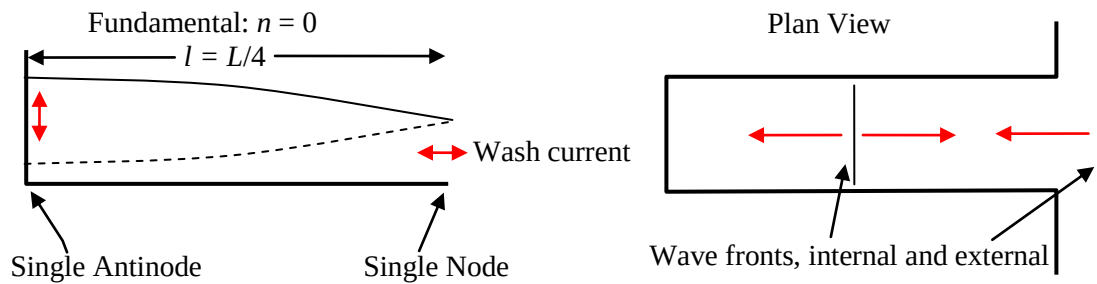
$$c = \frac{\omega}{k} = \sqrt{gy} \quad \text{where} \quad \omega = \frac{2\pi}{T} \quad \text{and} \quad k = \frac{n\pi}{l}$$

Thus
$$\frac{2\pi l}{n\pi T} = \sqrt{gy}$$

Or
$$T = \frac{2l}{n\sqrt{gy}} \quad \text{Characteristic oscillation periods in a closed basin of length } l \text{ and depth } y$$

 (Merian's Equation)

A similar derivation can be done for a harbour open at one end, with the possibility of waves of period T forcing the characteristic modes of oscillation:



The $n = 0$ mode is known as *Helmholtz mode* after its discoverer, or *pumping mode* after the effect it has on wash currents at the harbour mouth. For harbours and narrow natural inlets, this mode predominates. The characteristic period is:

$$T_o = \frac{4l}{\sqrt{gy}} \quad \text{Helmholtz (pumping) Period of Oscillation in an open harbour (s)}$$

Or $T = \frac{4l}{(2n+1)\sqrt{gy}}$ for higher oscillation modes in an open harbour

The maximum flow velocity at a harbour mouth in pumping mode is given by:

$$V_{max} = A\sqrt{g/y}$$

Harbour design begins with an analysis of onshore wave data to identify statistical distributions of wave periods. Harbour dimensions are then specified to avoid the most common waves. Theoretical methods, in particular the simple models set out above, are *indicative only* and should proceed with an experimental approach utilising scale models of various sizes forced by waves generated to scale the wave data.

It is not possible to design all possible seiche waves out of any particular harbour design. A recurrence interval approach should be used, whereby the characteristic seiche periods are designed to recur less than 100 to 200 years on average.

An example in Chadwick and Morfett (1993) sets out some general principles for the design of a scale model harbour. For similitude of long and short surface waves the Froude number is used with either depth or wavelength. In addition for similitude of tidal and other long wavelength oscillatory flows the Strouhal number is required:

$$\begin{aligned} Fr &= \frac{c}{\sqrt{gy}} & \text{where } c &= \sqrt{gy} & \text{Long waves, } \frac{L}{y} > 20 & \text{(For intermediate waves use the full hyperbolic equation)} \\ Fr &= \frac{c}{\sqrt{gL}} & \text{where } c &= \sqrt{gL/2\pi} & \text{Short waves, } \frac{L}{y} < 2 \\ St &= \frac{cT}{L} & \text{where } c & \text{is as determined above.} \end{aligned}$$

Example: For a 1:120 scale model harbour of design (full scale) depth 20 m, estimate the wave period required for simulation of a) the 2m tide and b) rough short waves (*chop*) 25 m long:

The model harbour is $20/120 = 0.167$ m deep. The model tide is 0.0167 m and the chop is 0.208 m long. (This is greater than the 1.7 cm needed to avoid surface tension effects ✓)

a) A tide is regarded as a long wave, so the conventional Froude number is used. Assuming a Froude ratio of 1 for similitude, the celerity ratio is

$$c_r = Fr_r \sqrt{g_r y_r} = 1 \times \sqrt{\frac{1}{120}} = 0.091$$

Similarly assuming a Strouhal ratio of 1, the period ratio is

$$T_r = \frac{St_r L_r}{c_r} = \frac{1 \times \frac{1}{120}}{0.091} = 0.091$$

The normal tidal period is $T_p = 12$ hours. The equivalent model period is thus

$$T_m = T_r T_p = 0.091(12) = \underline{1.095 \text{ hours}}$$

b) A 25 m short wave has a period of $T_p = L/c = 25/c$ where

$$c = \sqrt{\frac{gL}{2\pi}} = \sqrt{\frac{9.8(25)}{2\pi}} = 6.244 \text{ m/s}$$

So $T_p = 25/6.244 = 4.004 \text{ s}$

The same length ratio applies as before, regardless of whether we use length or depth in the celerity, so the Froude and Strouhal ratios yield the same period ratio:

$$T_r = 0.091 \quad \text{and so}$$

$$T_m = T_r T_p = 0.091(4.004) = \underline{0.364 \text{ seconds, a frequency of } 0.364^{-1} = 2.75 \text{ Hz}}$$

Statistical Wave Analysis

Since ocean waves are not constant in height or length it is essential to apply statistical techniques in the description of any given “sea”. The root-mean-square or RMS method is often used as an averaging technique:

eg applied to wave heights,

$$H_{RMS} = \sqrt{\frac{1}{N} \sum_{i=1}^N H_i^2} \quad \text{Where } N \text{ is the number of waves in the data series and } i \text{ is a counter}$$

Since wave power is proportional to H^2 this makes a lot of sense.

Another common average is the *significant height*, defined as the average height of the top 1/3rd of waves in the data:

$$H_{SIG} = H_{1/3} = \frac{1}{N/3} \sum_{j=1}^{N/3} H_j \quad \text{Significant wave height}$$

Similar ideas apply to a significant wave period:

$$T_{SIG} = T_{1/3} = \frac{1}{N/3} \sum_{j=1}^{N/3} T_j \quad \text{Significant wave period}$$

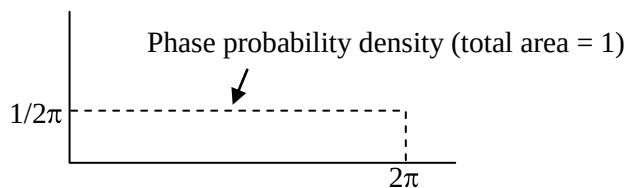
Beyond simple averages, it is possible to use Fourier analysis to break a linear wave series into its component frequencies. The idea is that linear sine waves of many different characteristics add together in a linear way to produce complex wave forms:

$$\eta(x, t) = \sum_{i=1}^m A_i \cos\left(\alpha_i - \frac{2\pi t}{T_i}\right)$$

Where A_i , α_i and T_i are the amplitude, *phase* and period of the i^{th} component of the wave form. These combined waves behave mathematically very much as linear ocean waves, including their tendency to travel in groups of celerity half the typical wave component celerity. So long as an ocean wave set is linear it can be decomposed analytically to find the wave components.

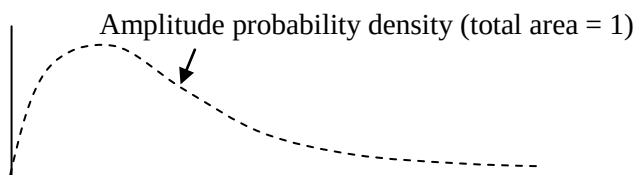
The phase α of a wave component is used rather than the non-dimensional x/L because of the freedom for different waves to have different origins and therefore different fundamental values of x . They are regarded as occurring at different phases, that is different points of trough or peak, in their cycle when they interact to combine additively in the wave form.

For any frequency, the phase is assumed to be uniformly distributed as a *spectrum* of phases between 0 and 2π , giving a probability density of $P(\alpha_i) = 1/2\pi$:



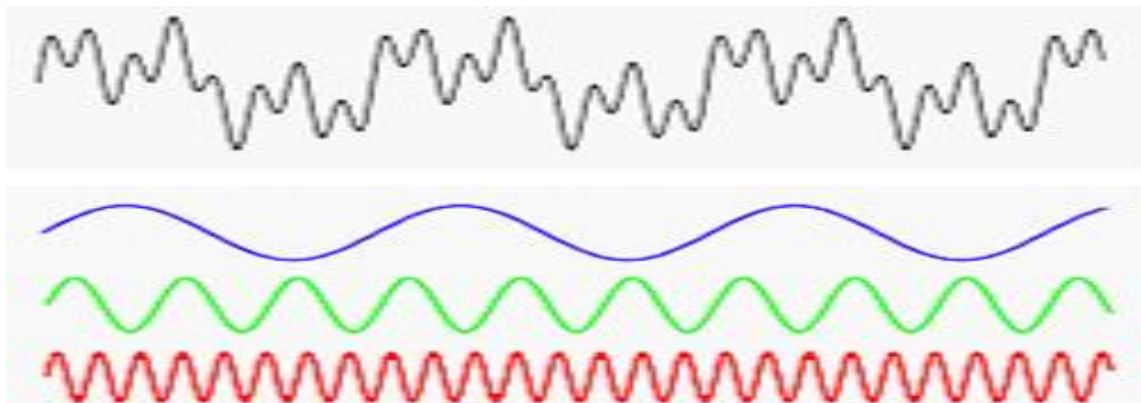
Similarly for any frequency the amplitude is distributed according to a *Rayleigh spectrum*:

$$P(A_i) = \frac{\pi}{2} \frac{A_i}{\mu_i^2} e^{-\left(\frac{\pi A_i^2}{4\mu_i^2}\right)}$$



Where μ_i is the mean of the i^{th} amplitude.

Wave frequencies are regarded as densely distributed (ie there are very many of them) between typical ocean frequencies of 0.05 to 1.0 Hz. For each frequency a distribution of amplitudes and phases is assumed. A wave series can be analysed to derive which combinations are most likely.



Fourier analysis: a complex wave (upper) can be analysed into simple sinusoidal components (three principal components shown). Fourier analysis is outside of the scope of a Fluid Mechanics course.

Non-linear Waves

Waves with either very long wavelength or high steepness ratio H/L are non-linear, ie they contain internal feedback mechanisms that produce behaviours that depart from the linear (Airy) theory described above. Boussinesq (1872) was the first to formulate a theory describing these waves, based on the vertical flow structure of horizontal and vertical flow velocities.

Other non-linear wave models include Saint-Venant, Stokes, cnoidal and dnoidal wave theory. The Saint-Venant method for flood waves is covered in the next section.

Oceanographic wave models routinely use non-linear computer simulations incorporating detailed sea-floor geography for wave prediction. This is highly specialist theory and is not covered here.

Non-Linear River Waves: Flood Waves

Flood waves are driven by rainfall events in a catchment and as such have only as many crests as there are distinct storm bursts in the rainfall. An exception to this can occur where catchments of different size meet at a confluence: The runoff from the smaller catchment travels more quickly than the larger catchment, so the flood wave arrives at the confluence in two distinct phases. The smaller catchment may even suffer a double flood event, first due to runoff from its own catchment, and second due to backwash from the larger flood wave. This actually occurred in the Coffs Harbour Regional Museum in 1998 and was a mystery until someone from UNE came and explained it!

Flood routing methods can be classified as hydraulic – in which continuity and dynamic equations are used – or hydrologic, which generally use the continuity equation alone. The following pages are intended to augment your studies in hydrologic flood routing, on the basis that hydraulic methods are more suitable for modelling complex runoff phenomena such as the CHRM flood.

Kinematic Waves

A kinematic wave is a hydrologic flood wave occurring on a plane surface of known roughness. It is often used in design calculations for medium sized individual catchment elements feeding into a tributary stream. Smaller catchments, such as housing allotments draining into a street gutter, are typically handled by standard entry times such as 5 minutes per typical suburban house block. Larger catchments are treated as composites of kinematic or other types of routing wave elements.

A kinematic wave shows the development of runoff from a rainfall event of constant intensity I , usually stated in mm/hour. The rainfall is a constant inflow rate which can be converted to m^3/s as:

$$Q_i = I.A / 360 \quad \text{Rainfall inflow rate (m}^3/\text{s)}$$

Where A is the area in hectares ($1 \text{ ha} = 10,000 \text{ m}^2$). The factor 360 converts time to seconds and volume to m^3 from the inputs of mm/h and ha.

Infiltration into the soil and storage of standing water in hollows on the surface reduces the runoff-producing component of the rainfall by an uncertain amount. In practice it is usual to apply a runoff coefficient $C < 1$ which depends on the percentage of impervious area in the surface, ie its state of development (you'll see values of C tabulated in your hydrology courses):

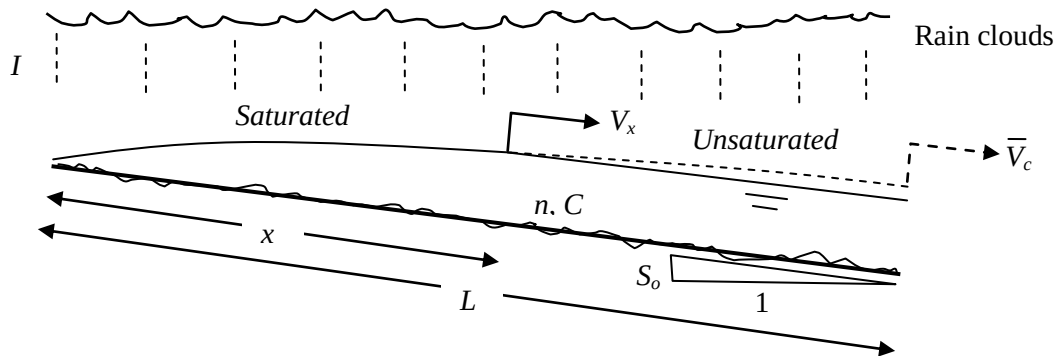
$$Q_r = C.I.A / 360 \quad \text{Maximum runoff flowrate (m}^3/\text{s)}$$

The actual, time-dependent (as opposed to maximum) outflow from the catchment depends on the depth of flow at the outfall, which starts out at zero. Since the outflow is less than the rainfall inflow, the catchment “fills up” until the depth is sufficient to drive an outflow equal to the inflow. This takes a certain amount of time, known as the *time of concentration* of the catchment. Based on Manning's equation:

$$t_c = 6.94 \frac{(L.n)^{0.6}}{I^{0.4} S_0^{0.3}} \quad \text{Concentration time (minutes)}$$

Where L is the length of the catchment (m), S_0 is the slope (m/m) and n is the Manning's roughness. The average celerity of the flood wave is thus:

$$\bar{c} = L / 60t_c = \frac{(I.L)^{0.4} S_0^{0.3}}{416n^{0.6}} \quad \text{Mean kinematic wave celerity (m/s)}$$



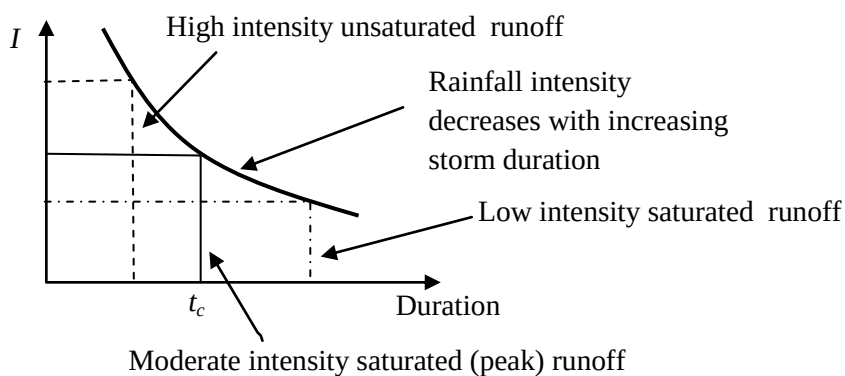
Schematic of Kinematic Wave. The saturated region x has steady flow depth, while the unsaturated region is unsteady. The wave moves downhill producing peak runoff when the catchment is "full".

Since the saturated region is fully developed the intermediate length x is analogous to a fully developed subcatchment with its own concentration t_{cx} and wave celerity given by:

$$c_x = x / 60t_{cx} = \frac{(I.x)^{0.4} S_0^{0.3}}{416n^{0.6}} \quad \text{Kinematic wave celerity (m/s)}$$

Thus a kinematic flood wave accelerates nearly in proportion to $\sqrt{x}$ as it moves downhill.

The maximum theoretical runoff from a plane surface occurs when the rainfall duration is equal to the time of concentration: Short storm bursts are more intense than longer storm systems. A storm of duration shorter than t_c does not fully develop the catchment, while a storm longer than t_c has a lower intensity and so produces a smaller runoff flowrate.



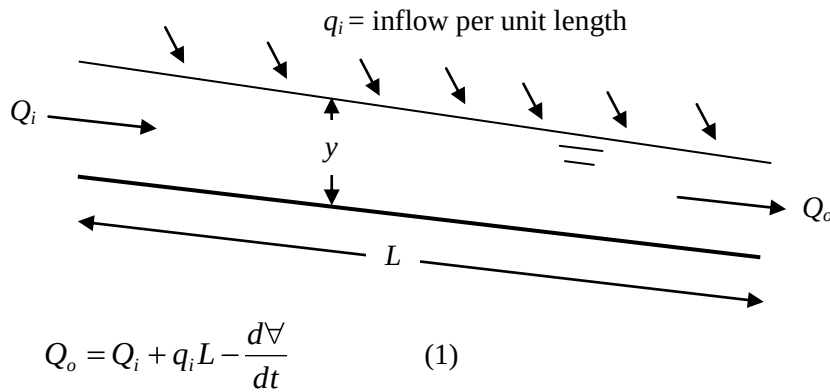
The process of calculating peak runoff from a catchment is iterative: calculate t_c for an assumed intensity (say 100 mm/h), then look up the local Intensity-Frequency-Duration charts (IFD) for the intensity associated with that t_c , and recalculate t_c . When this converges, calculate the maximum runoff flowrate Q_r as above.

Unsteady Gradually Varied Flood Waves: The Saint-Venant Equations

An open channel with changing upstream and lateral inflow rates (due to regional and local catchment runoff) can be modelled via a system of linked continuity and momentum equations known as the *Saint-Venant Equations* after the mathematician A.J. de Saint-Venant who first derived them. The equations are non-linear and require a computational approach for solution.

Saint-Venant Continuity

Any difference between inflow and outflow over a channel reach¹ must be accompanied by a change in storage, reflected in a change in average flow height in that reach.



V is the volume of water instantaneously contained or *stored* in the reach. Any imbalance between outflow Q_o and the two sources of inflow Q_i and $q_i L$ must be balanced by an increase or decrease in storage, hence a rate of volume change.

Over a short reach, the depth y and width B of the flow are approximately constant so

$$V = BLy$$

Thus
$$\frac{dV}{dt} = BL \frac{dy}{dt}$$

We now have a rate of change of flow depth for short lengths. In the limit as L becomes so short it can be treated as an infinitesimal dx , we get:

$$\frac{dV}{dt} = Bdx \frac{dy}{dt}$$

Over the infinitesimal dx the flowrate Q changes from Q_i to Q_o as:

$$Q_o = Q_i + \frac{dQ}{dx} dx$$

Thus equation (1) becomes

$$Q + \frac{dQ}{dx} dx = Q + q_i dx - Bdx \frac{dy}{dt}$$

Cancelling and dividing by dx , we arrive at:

¹ A short length of channel is known as a *reach*

$$\frac{dQ}{dx} + B \frac{dy}{dt} = q_i$$

Now since $Q = V.A$, applying the chain rule of differential calculus:

$$V \frac{dA}{dx} + A \frac{dV}{dx} + B \frac{dy}{dt} = q_i \quad (2) \quad \text{(Unsteady Continuity Equation)}$$

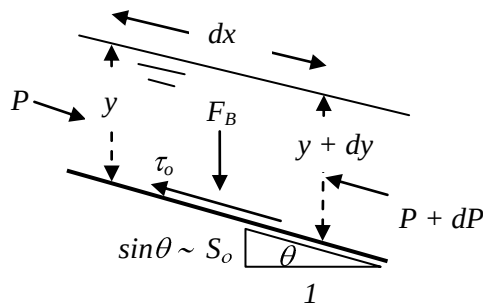
So the rate of change of flow depth can be calculated by the changes in flow area and velocity between two points. Over a short time interval this gives a new flow depth, which in turn gives new values of flow area and velocity, which can be used to calculate the change rate in the next time step.

In fact this is an incomplete specification as the changing depth and flow velocity create force and momentum reactions which must be allowed for. A momentum equation is needed:

Saint-Venant Momentum

The principal forces acting on a fluid body are friction (τ_o), body forces due to weight (F_B) and pressure forces (P). Any imbalance between these forces produces a change in fluid momentum:

$$\sum F_x = \frac{d(MV)}{dt} \quad \text{an equivalent expression of the Force-Reaction law}$$



An element of channel of width W ,
Cross-sectional Area $A = Wy$ and
Friction Surface Area $A_s = (W + 2y) dx$

The slope is small enough to simplify
some angular terms eg $S_o = \tan \theta \sim \sin \theta$

Work out all the forces in the downslope (x) direction:

Downslope Weight: $F_B \sin \theta = \rho g \forall \sin \theta = \rho g A dx S_o$

Downslope Friction: $-\tau_o A_s = -\tau_o P_w dx$ $P_w = \text{wetted perimeter} = W + 2y$

Downslope Pressure: $P - (P + dP) A = -dP A = -\rho g dy A$

Hence the total downslope force is

$$\sum F_x = \rho g A dx S_o - \tau_o P_w dx - \rho g dy A$$

This equates to the change in momentum, including local and convective accelerations:

$$\frac{d(MV)}{dt} = \frac{\partial(MV)}{\partial t} + \frac{\partial(MV)}{\partial x} \frac{\partial x}{\partial t} = \frac{\partial(\rho A dx V)}{\partial t} + \frac{\partial(\rho A dx V^2)}{\partial x} V$$

So $\rho g A dx S_o - \tau_o P_w dx - \rho g dy A = \frac{\partial(\rho A dx V)}{\partial t} + \frac{\partial(\rho A dx V^2)}{\partial x}$

Divide both sides by ρdx :

$$g A S_o - \frac{\tau_o}{\rho} P_w - g \frac{dy}{dx} A = \frac{\partial(AV)}{\partial t} + \frac{\partial(AV^2)}{\partial x}$$

The RHS can be differentiated by the chain rule:

$$\frac{\partial(AV)}{\partial t} + \frac{\partial(AV^2)}{\partial x} = A \frac{dV}{dt} + V \frac{dA}{dt} + V^2 \frac{dA}{dx} + 2VA \frac{dV}{dx}$$

Rearrange this to isolate terms equivalent to the continuity equation:

$$A \frac{dV}{dt} + V \frac{dA}{dt} + V^2 \frac{dA}{dx} + 2VA \frac{dV}{dx} = A \frac{dV}{dt} + VA \frac{dV}{dx} + V \left(\frac{dA}{dt} + A \frac{dV}{dx} + V \frac{dA}{dx} \right)$$

But $\frac{dA}{dt} = B \frac{dy}{dt}$

So the last term on the RHS reduces to

$$V \left(B \frac{dy}{dt} + A \frac{dV}{dx} + V \frac{dA}{dx} \right) = Vq_i \quad (\text{using equation (2) above})$$

Thus $g A S_o - \frac{\tau_o}{\rho} P_w - g \frac{dy}{dx} A = A \frac{dV}{dt} + VA \frac{dV}{dx} + Vq_i$

Divide both sides by gA:

$$S_o - \frac{\tau_o P_w}{\rho g A} - \frac{dy}{dx} = \frac{1}{g} \frac{dV}{dt} + \frac{V}{g} \frac{dV}{dx} + \frac{Vq_i}{gA}$$

But $\frac{\tau_o P_w}{\rho g A} = \frac{\tau_o}{\rho g R} = S_f$ (Good old friction slope equation)

So we arrive at:

$$S_o - S_f - \frac{dy}{dx} = \frac{1}{g} \frac{dV}{dt} + \frac{V}{g} \frac{dV}{dx} + \frac{Vq_i}{gA}$$

Or $S_f = S_o - \frac{dy}{dx} - \frac{V}{g} \frac{dV}{dx} - \frac{1}{g} \frac{dV}{dt} - \frac{Vq_i}{gA}$ (3) Unsteady friction slope equation

The unsteady friction slope equation incorporates everything we need to know about unsteady fluid momentum in an open channel and is used in conjunction with the unsteady continuity equation to numerically solve not only flood waves but many other types of shallow river wave problems.

This represents a series of approximations:

$$S_f = S_o \quad \text{Steady, uniform flow (Manning's equation)}$$

$$S_f = S_o - \frac{dy}{dx} - \frac{V}{g} \frac{dV}{dx} \quad \text{Steady, non-uniform (Standard Step Method)}$$

$$S_f = S_o - \frac{dy}{dx} - \frac{V}{g} \frac{dV}{dx} - \frac{1}{g} \frac{dV}{dt} - \frac{Vq_i}{gA} \quad \text{Unsteady, non-uniform (Saint-Venant equations)}$$

As with the standard step method (see Open Channel Hydraulics) the friction slope is identified with Manning's equation:

$$S_f = \left(\frac{nQ}{AR^{2/3}} \right)^2$$

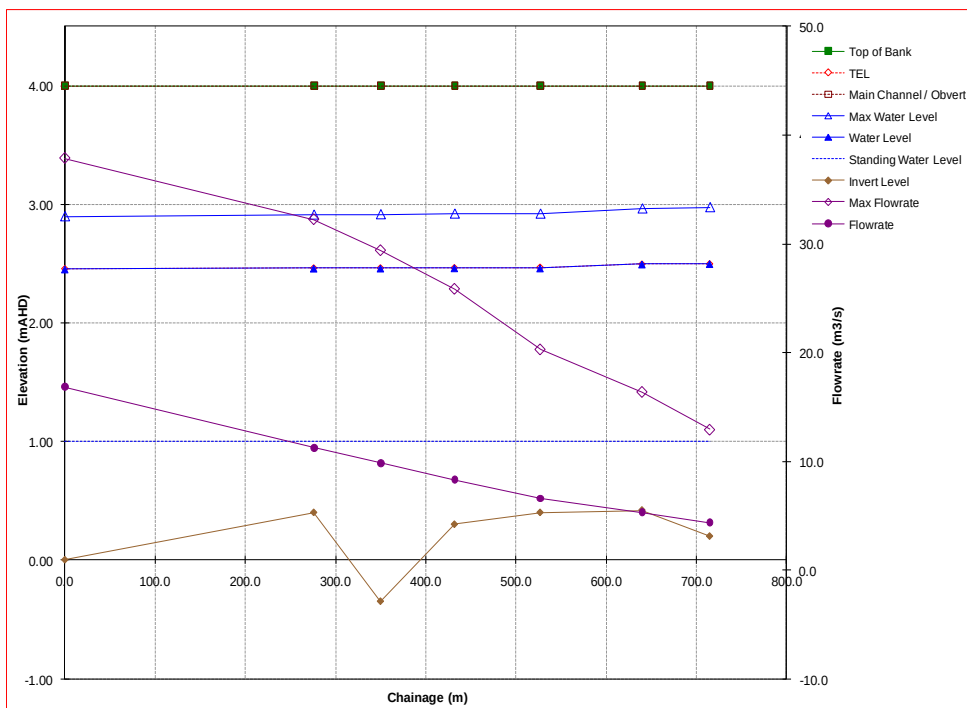
$$\text{so} \quad \left(\frac{nQ}{AR^{2/3}} \right)^2 = S_o - \frac{dy}{dx} - \frac{V}{g} \frac{dV}{dx} - \frac{1}{g} \frac{dV}{dt} - \frac{Vq_i}{gA}$$

are the Saint-Venant Equations

$$\text{and} \quad q_i = V \frac{dA}{dx} + A \frac{dV}{dx} + B \frac{dy}{dt}$$

The method of solution is an engineering technique that requires professional supervision rather than an academic tutorial, however in general terms the method is:

- Obtain cross-section survey data of the channel at approximately evenly spaced stations, including offset, elevation and roughness data to allow calculation of area, wetted perimeter, and conveyance.
- Obtain plan survey data of catchment slope, area, roughness and hydraulic connectivity including natural and constructed drainage surfaces and pathways.
- Obtain hydrologic climate data for the regional location, including rainfall intensity-frequency-duration curves and rainfall hyetographs suited to the design question.
- Obtain other design discharges such as industrial outfalls.
- Obtain hydraulic boundary conditions such as tide levels and recorded outfall flow levels for the design circumstances.
- Identify initial ($t = 0$) flow conditions along the channel including flow rate and height. Set these out as a one-dimensional space array of hydraulic boundary conditions for $t = 0$.
- Connect design hydrologic flowrates as functions of time to the channel flowrates at appropriate inflow points (“connect” in the sense of ensuring formulae are looking in the correct digital location for data eg spreadsheet rows, columns and cells).
- Beginning with $t = 0$, isolate time rates of change for depth and velocity and evaluate these for the known initial conditions at each location in the space array. Multiply the time rates of change by a small time interval Δt to find the new values of depth and velocity at $t = 0 + \Delta t$.
- Calculate friction slopes and other variables relevant to the hydraulic solution of other variables at $t = 0 + \Delta t$.
- Check that the first incremental solution is convergent and provides reasonable values for all terms.
- Repeat for $t = 2\Delta t, 3\Delta t$, and so on, until a complete flood wave profile is found.



Snapshot of flood wave in Carrall's Creek, Coffs Harbour, simulated by S-V equations. The graph shows flood levels and flowrates compared to their maximum values previous to this time. Although the flowrate is dropping rapidly the flood water level persists due to continuing flooding in Coffs Creek. A second floodwave from the main channel follows shortly after this simulation point.

Method of Characteristics

The Saint-Venant equations are analytically soluble for only the simplest problems, which are thus often used as first estimates of more complex problems. For example the classic calculation of a frictionless dam-break or sudden sluice gate operation is similar to the full solution (and can be done in a hurry!)

The *method of characteristics* is applied not only in fluid mechanics but in many problems requiring an approximate solution of harmonic DE's. In fluid mechanics the mathematical tools of the method correspond directly to physical interpretations, which I will try to emphasise.

The basic assumptions are a rectangular channel ($B = \text{constant}$) with zero inflows ($q_i = 0$). The first step is to make a substitution for the flow depth y from the river wave celerity:

$$c = \sqrt{gy}$$

$$\text{Thus } y = \frac{c^2}{g} \quad \text{and} \quad dy = 2\frac{c}{g}dc$$

Substituting these into the continuity and momentum equations with $q_i = 0$ we obtain:

$$2V \frac{\partial c}{\partial x} + c \frac{\partial V}{\partial x} + 2 \frac{\partial c}{\partial t} = 0 \quad \text{Continuity}$$

$$\text{And } 2c \frac{\partial c}{\partial x} + V \frac{\partial V}{\partial x} + \frac{\partial V}{\partial t} = S_o - S_f \quad \text{Momentum}$$

I've switched to partial differentials to highlight the fact that we will shortly be relying more closely on the distinctions between space and time derivatives. Certain symmetries are starting to emerge. If we add the two equations together we get:

$$\frac{\partial(V + 2c)}{\partial t} + \frac{\partial(V + 2c)}{\partial x}(V + c) = S_o - S_f \quad (4)$$

And if we subtract momentum from continuity we find:

$$\frac{\partial(V - 2c)}{\partial t} + \frac{\partial(V - 2c)}{\partial x}(V - c) = S_o - S_f \quad (5)$$

The symmetries are now even more pronounced. I'll try to explain:

The momentum equation contains local and convective reactions, that is forces arising from accelerations because of unsteady flowrates AND because of the movement of flow through non-uniform flow fields. In either case a reaction to the fluid acceleration is produced. This combined force reaction can be derived mathematically from the chain rule:

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \frac{dx}{dt} = \frac{dV}{dt}$$

$\uparrow$ local $\uparrow$ convective $\uparrow$ total acceleration

The dx/dt term is the flow velocity. Comparing this to either of equations (4) or (5) above, the LHS clearly contains both local and convective terms, with dx/dt replaced by $(V + c)$ or $(V - c)$ respectively. The physical implication is that relative velocities of $V + c$ or $V - c$, that is of a shallow wave travelling downstream or upstream, are symmetrical as per equations (4) and (5).

A consequence of the mixed partial differential is that observers moving at either of these *characteristic* velocities effectively cancel the spatial change from their frame of reference, so they see only the temporal change. We are left with:

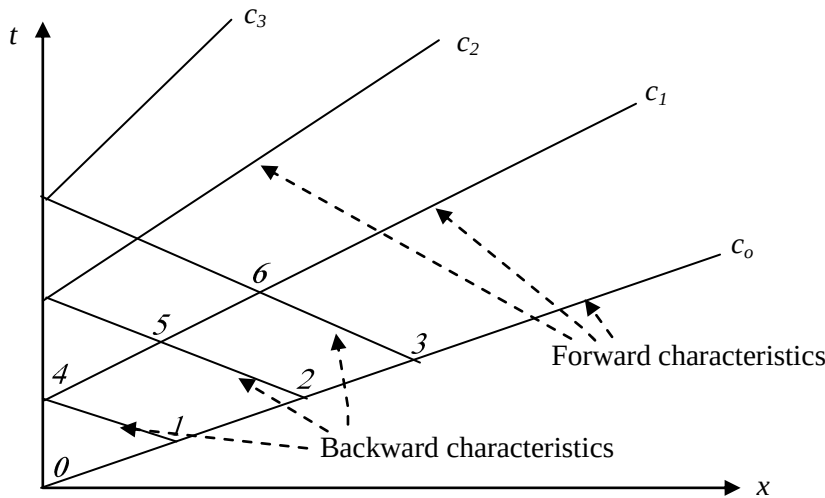
$$\frac{d(V + 2c)}{dt} = S_o - S_f \quad \text{for a reference frame moving at } \frac{dx}{dt} = V + c$$

$$\frac{d(V - 2c)}{dt} = S_o - S_f \quad \text{for a reference frame moving at } \frac{dx}{dt} = V - c$$

These are often simplified even further as:

$$\frac{d(V \pm 2c)}{dt} = S_o - S_f \quad \text{for a reference frame moving at } \frac{dx}{dt} = V \pm c$$

The method is to identify V and $c = \sqrt{gy}$ as boundary conditions along the length of a channel and use forward and backward characteristics (ie lines of travel at $V + c$ and $V - c$) to solve for subsequent velocities and celerities, and hence solve for flow depths as $y = c^2/g$.



Regions that cannot communicate by a wave in the available time are not effected by each other. For example the values of V and c at point 5 are influenced by points 0, 1, 2 and 4, but not by point 3 or 6.

Example. Assume we know the solution (V_3, c_3) and (V_5, c_5) at points 3 and 5 (after solving out from 0) and need to determine the solution at 6. A forward characteristic from 5 and a backward from 3 can be solved simultaneously for 6.

Along the forward characteristic:

$$d(V + 2c) = (V_6 + 2c_6) - (V_5 + 2c_5) = \Delta t (S_o - S_f)$$

Where S_o is from the survey data and S_f is determined by Mannings equation at 5.

Along the backward characteristic:

$$d(V - 2c) = (V_6 - 2c_6) - (V_3 - 2c_3) = \Delta t(S_o - S_f)$$

Again S_o is from the survey data and S_f is determined by Mannings equation at 3.

Adding these characteristic equations together we obtain

$$2V_6 + 2c_3 - V_3 - V_5 - 2c_5 = 2\Delta t(S_o + S_f)$$

$$\text{Or } V_6 = (c_5 - c_3) + \frac{(V_3 + V_5)}{2} + \Delta t(S_o + S_f)$$

Subtracting gives

$$4c_6 + V_3 + 2c_3 - V_5 - 2c_5 = 0$$

$$\text{Or } c_6 = \frac{(V_5 - V_3)}{4} + \frac{c_5 + c_3}{2}$$

This solves the values of V and c and hence the flow depth y at point 6. We only need to know the actual location of point 6 (in space and time):

The positive characteristic from 5 has a gradient $1/(V_5 + c_5)$ (as per $t = x/v$) and the negative characteristic from 3 has a gradient $1/(V_3 - c_3)$. The location of 6 in space and time is the intersection of these two gradients: From 5 to 6 there is a distance Δx_5 , and from 3 to 6 a *negative* distance Δx_3 .

There is a common time interval Δt from either 3 or 5 to 6:

$$\Delta t = \frac{\Delta x_5}{V_5 + c_5} = \frac{\Delta x_3}{V_3 - c_3} \quad (V_3 - c_3) < 0$$

$$\text{So } \Delta x_5 = \frac{\Delta x_3(V_5 + c_5)}{V_3 - c_3}$$

$$\text{But } \Delta x_5 + (-\Delta x_3) = x_3 - x_5$$

ie the sum of the two absolute distances is the distance between 3 and 5.

$$\text{So } \frac{\Delta x_3(V_5 + c_5)}{V_3 - c_3} - \Delta x_3 = \Delta x_3 \left(\frac{V_5 + c_5}{V_3 - c_3} - 1 \right) = x_3 - x_5$$

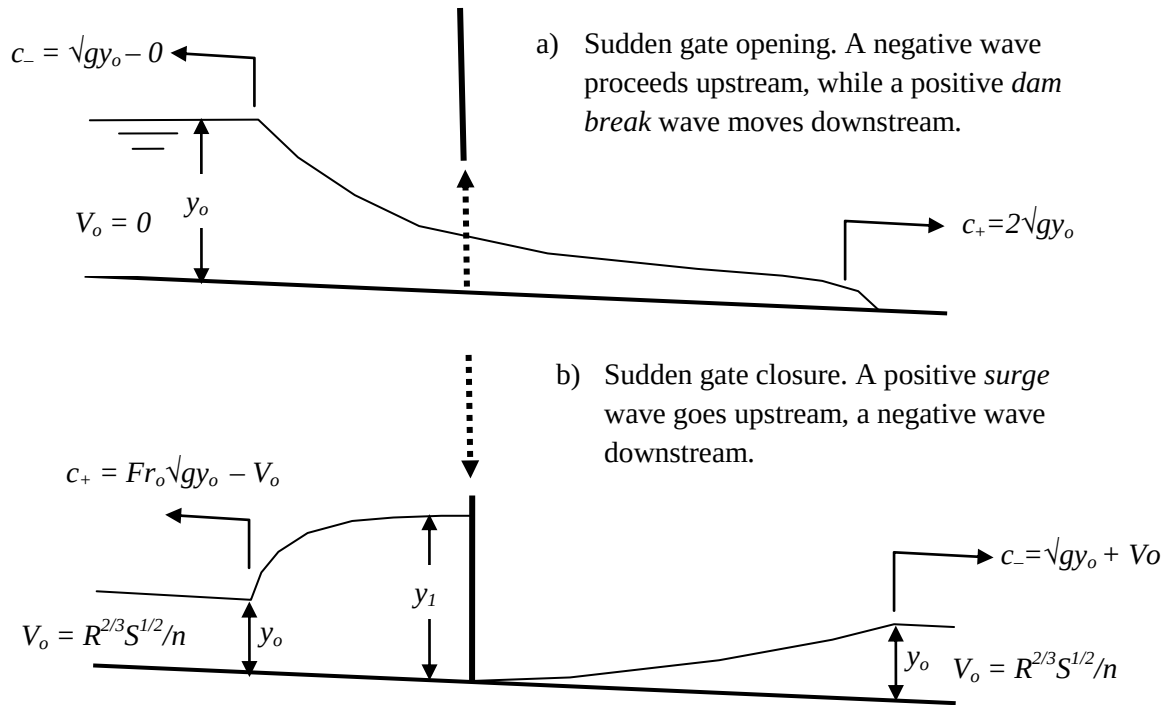
$$\text{and } \Delta x_3 = \frac{x_3 - x_5}{\left(\frac{V_5 + c_5}{V_3 - c_3} - 1 \right)} < 0$$

Δt can now be found, locating $x_6 = x_3 + \Delta x_3$ and $t_6 = t_3 + \Delta t$.

Frictionless Non-Linear River Waves

Sudden changes in a river surface profile can be triggered by rapid opening and closing of a sluice gate or (in disastrous circumstances) a break in a dam. In these circumstances the *information* that a gate has opened or closed must be transmitted upstream or downstream in the form of a wave. Points on the channel outside the travel time of the wave *do not know that the event has occurred*.

Two situations of gate open and closure are shown in the figure below:



Negative waves travel at the normal celerity of a river wave at the initial depth, plus or minus the initial flow velocity V_o .

A positive surge wave can be analysed as a rectangular bore:

$$\frac{y_1}{y_o} = \frac{\sqrt{1 + 8Fr_o^2} - 1}{2}$$

Where $Fr_o = \frac{V_o + c_+}{\sqrt{gy_o}}$ in the reference frame of the positive surge wave

And $y_1 = (V_o + c_+)^2 \frac{y_o}{g} + y_o$ (by momentum and pressure balance)

Rearranging to isolate c_+ :

$$Fr_o = \sqrt{\frac{\left(2\frac{y_1}{y_o} + 1\right)^2 - 1}{8}} = \frac{V_o + c_+}{\sqrt{gy_o}}$$

Hence $c_+ = Fr_o \sqrt{gy_o} - V_o$

Solution method (Positive surge wave):

- 1) Obtain y_o , S_o and n
- 2) Calculate R
- 3) Calculate V_o
- 4) Guess a value of c_+
- 5) Use y_o , V_o and c_+ to calculate y_1
- 6) Calculate Fr_o
- 7) Calculate c_+
- 8) Repeat from 5) to convergence.

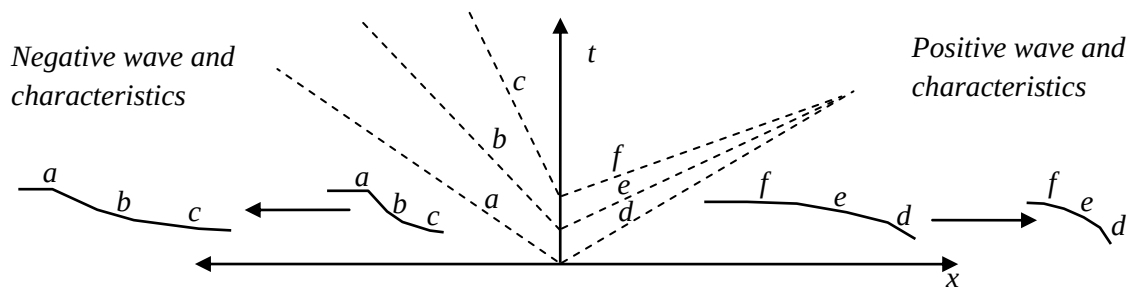
A positive break wave or dam break wave is complex and still the subject of research (eg Chanson, 2008) however a simple frictionless solution (Ritter, 1892) that closely matches experiment is:

$$c_+ = 2\sqrt{gy_0} \quad \text{Positive or Dam Break Wave Celerity (m/s)}$$

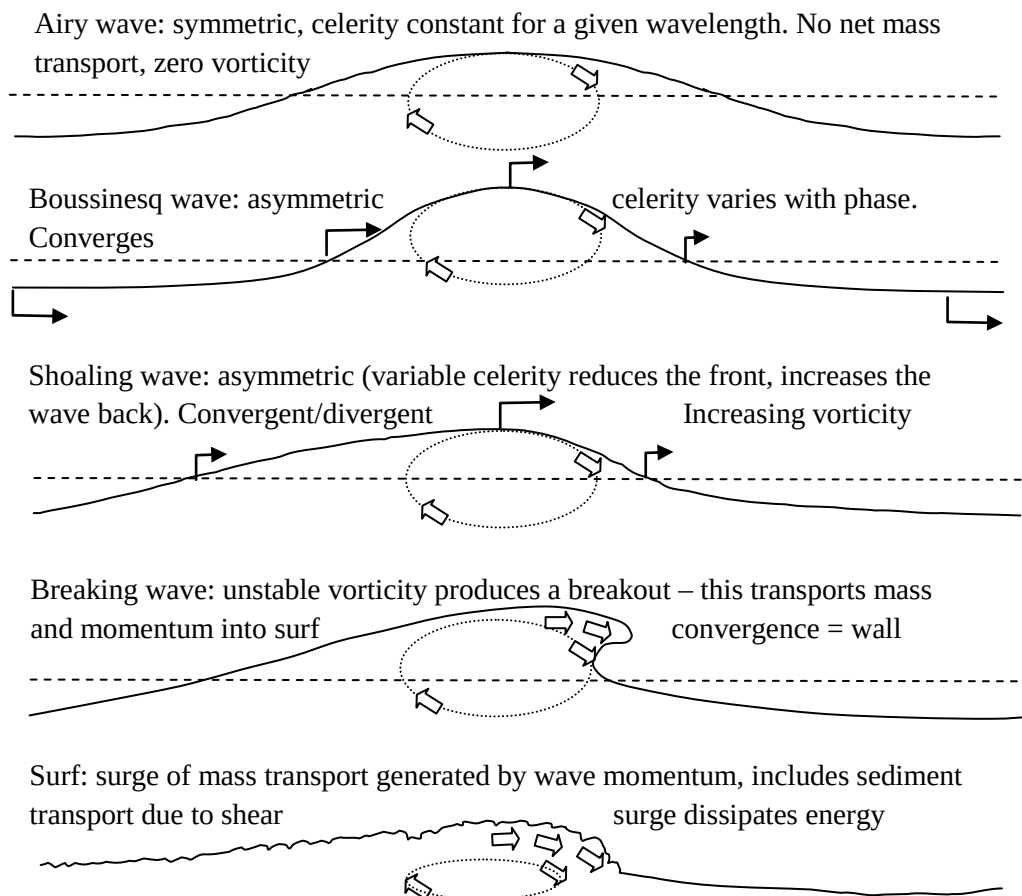
Convergent (positive) and Divergent (negative) waves

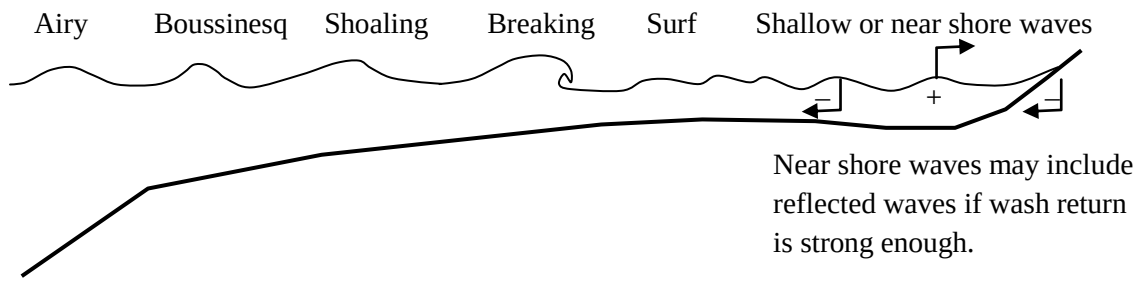
The surface profile of a negative waves rises in the direction of travel. The celerity of points on a negative profile therefore increase in the direction of travel, and the wave tends to elongate or diverge.

In contrast the surface profile of a positive waves falls in the direction of travel. The celerity of points on a positive profile therefore decrease in the direction of travel, and the wave tends to converge, leading to a “wall of water” which dissipates due to friction.

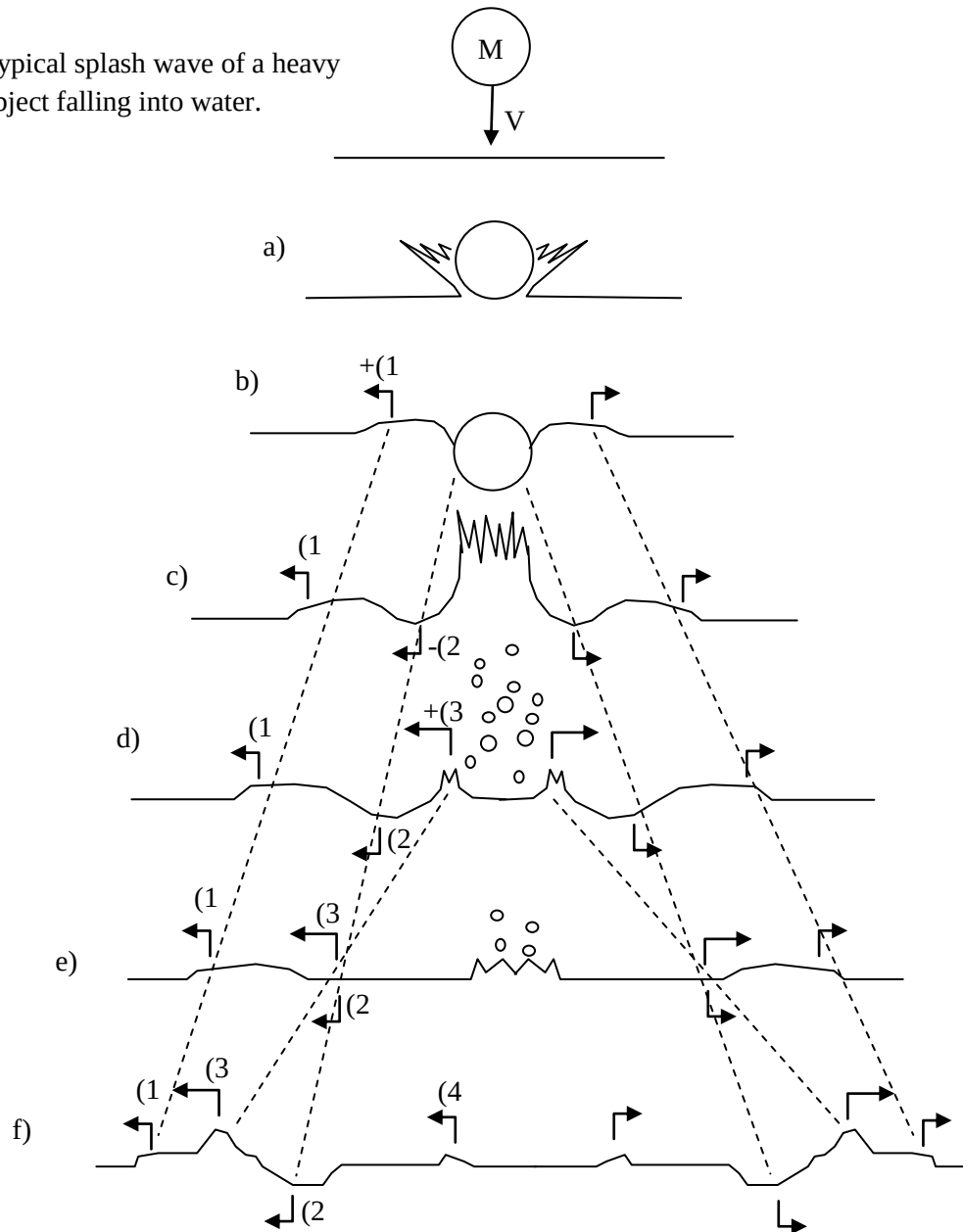


Characteristics are a good way of understanding waves of all kinds. Some typical ocean wave and shoaling formations are:





Typical splash wave of a heavy object falling into water.



- Initial jet splash.
- Sudden compression produces a positive displacement wave +(1)
- The displacement wave and its negative trough -(2) propagate outwards, the negative diverging and the positive converging. A central burst splash occurs.
- Burst splash produces a strong nonlinear splash wave +(3) which is faster than (1) and (2)
- The splash wave and displacement trough combine destructively, cancelling out to produce a long node that appears to flatten the wave. Fallout from the burst splash patters down.
- The displacement and splash waves combine constructively, producing a surge. Fallout produces a small kinematic wave (4)

Large Ocean Waves

Tsunami

Tsunami (Japanese for “harbour wave”) are destructive waves generated by seismic displacements – earthquakes or landslides – occurring in the sea bed. Tsunami have very high celerity and long wavelength, but almost imperceptible height until shoaling occurs near the shoreline. Assuming the cause is a single seismic event, a tsunami is initially a solitary wave or *soliton*, but the non-linearity of the wave causes dispersive actions that break the soliton into a train of intermediate-length waves.

As a tsunami approaches a coastline, shoaling effects coupled with friction and nonlinearities reduce the celerity and wavelength even further. Their arrival onshore is usually preceded by a dramatic trough phase in which the sea level falls as a large volume of water is drawn out to sea, before returning in an extreme crest phase.

Tsunami reach heights of several metres as they break, and thereafter they tend to behave as a series of mounting surges, driven inland by large flowrates and high momentum forces. Tsunami collect debris as they go and usually dissipate from a wall of water to a sheet of mud and wreckage.

Since the Indian Ocean tsunami on Boxing Day 2004, a large international effort has invested considerable funds in tsunami detection worldwide.

Freak or Rogue Waves

Ocean waves of the order of 30 to 40 m are occasionally reported, appearing seemingly out of nowhere in otherwise typical seas. At one time their rarity meant that they were thought to be a myth, but large-scale oceanography has confirmed the existence of a transient wave phenomenon which matches these reports.

A rogue wave is a wave greater than twice the significant wave height, usually isolated ie with no similarly large waves in the group, and with a trough of normal depth preceding the very high wave crest. Rogue waves are transient in both time and space – an isolated wave crest appears for no apparent reason, persists for a relatively short time (of the order of several minutes to a few hours) and then disappears again. This explains their apparent rarity – they are only encountered if you happen to be in the wrong place at the wrong time.

At least four mechanisms have been proposed for the occurrence of rogue waves:

1. Linear superposition of waves, varying according to the Rayleigh distribution of amplitudes. Since ocean waves travel at different celerities it must occasionally happen that a number of high energy waves combine at a point, producing a high local amplitude. There are a number of problems with this description, in particular the absence of similarly large waves travelling in the group.
2. Interactions of waves with ocean currents, causing amplification and focussing of wave amplitudes (this is not well understood).
3. Crossing sea interactions – waves from different storm events interact due to diffraction behind islands. This by itself does not explain the occurrence of rogues in the deep ocean far from any islands.
4. High-order non-linear properties of long waves. Complex variable analyses suggest that rogue waves can emerge from a normal sea with none of the above interaction effects – ie they are self-generating due to a quasi-resonance of individual wave groups. This is currently the preferred and most likely mechanism.

Tides

Although strictly speaking an astrophysical phenomena, the tides can be regarded as the world's longest possible gravity wave, consisting of two cycles that encompass the earth in 24 hours. The tides are generated by the gravitational interaction of the oceans with the moon *and* to a lesser extent the sun:

- High tides occur simultaneously on opposite sides of the Earth. The high tide on the side nearest the moon is due to the dominance of gravitational force on that side, while the opposite high tide is due to the dominance of the centrifugal force of the earth-moon orbit on the side furthest from the moon.
- Spring (peak high) tides occur around twice a month when the sun and moon are in gravitational alignment, ie at the full or new moon, while neap (peak low) tides occur when the sun and moon are out of alignment, ie at quarter waxing or waning moons.
- The Proxigean tide is an unusually high tide caused by a new moon (moon and sun on same side of the earth) at the point of nearest moon approach (perigee). This occurs approximately once every 18 months.
- Latitudinal differences in tide heights are due to the varying angle of azimuth to the moon and sun, varying with the seasons. The equator has relatively low tides compared to locations at 23.5° N or S, particularly at spring or autumn equinoxes.

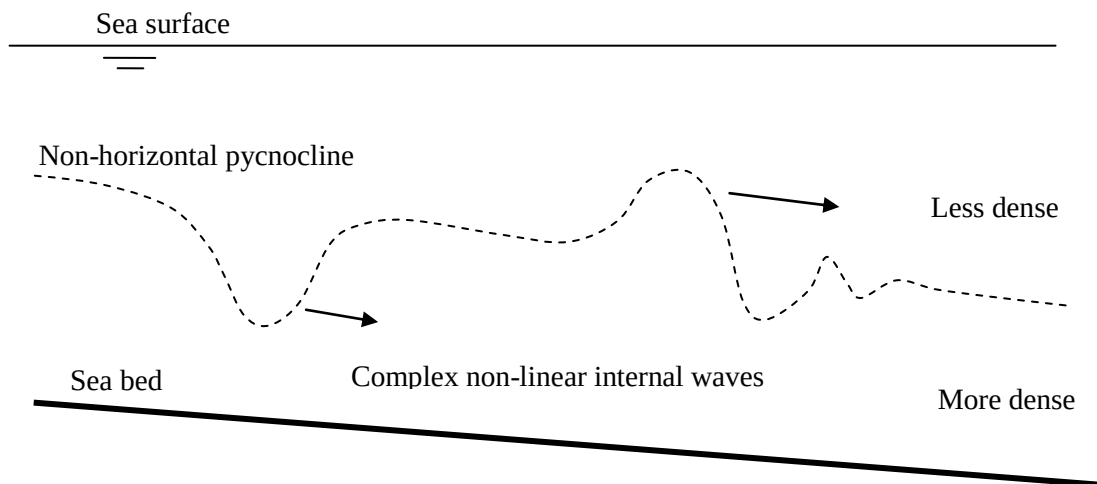
Because of the very high analytic reliability of celestial orbits, the tides can in principle be predicted any amount of time in advance. Coastal interactions are a complicating factor, so the accuracy of predicted tide heights can vary from place to place. Because their wavelength is far greater than their depth they are regarded as shallow waves and as such are highly sensitive to shoaling: as a result tide height depends on coastal depth. For example the very high tides on the northern coastline of Australia are due to the very broad and shallow coastal shelf.

Tides differ from shallow waves in that they travel at enormous celerity (the earth's circumference in 12 hours) and are responsible for large-scale ocean currents that extend across the face of the earth. Tides are important for ecological reasons and are significant forces in estuarine geomorphology and in the formation of twice-daily riverine bores, such as the Ch'ient'ang'kian bore in China which reaches heights of up to 7 metres.

Internal Waves

Internal waves occur at the interface between fluids of differing density, such as at *haloclines* (saline to freshwater gradients) or *thermoclines* (warm to cool water). A general density gradient caused by combined thermal and saline density differences is known as a *pycnocline* and may occur at any depth. Fluids of different density do not mix easily, so there is a distinct and stable surface that internal waves follow.

Internal waves can range in wavelength from a few metres to tens of kilometres. They do not necessarily orient to the horizontal, since they follow the slope of the pycnocline. They are highly nonlinear, with asymmetric crests and troughs. They are often visible from aircraft or satellites as an apparent difference in ocean texture or streaks on the surface.



Atmospheric *wave clouds* are also a type of internal wave between air layers of different density due to vapour pressure and temperature.

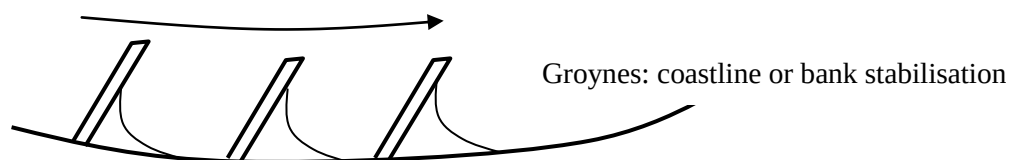
Oceanic internal waves are generally slow-moving and frequently occur in isolation as single waves or solitons. They are often visible on the surface as bands of alternating rough and smooth water or at shallow depths as they reflect and refract light. The celerity of internal waves decreases with the difference in density.

Although slow-moving, internal waves can carry a considerable amount of energy. A phenomenon known as *dead water* arises where a pycnocline occurs at the depth of a ship's keel: the ship generates internal waves which carry away large amounts of energy, creating drag and slowing the ship. Internal waves are usually not visible from the deck of a ship, so this phenomenon was a source of mythology or at least mystery for many centuries.

Internal waves are the subject of ongoing research and mapping. The theory governing their celerity and fluid displacement is complex and not yet fully established.

Wave Damping and Sea Defence Structures

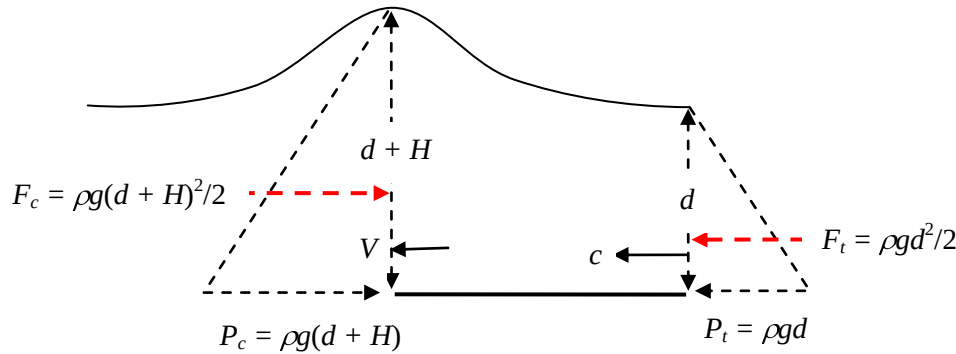
A number of structural alternatives are now available for the control of waves and the protection of shorelines. These include submerged and floating structures which may be permeable or semi-permeable. Sea walls may be wide or narrow crested, with a range of face slopes to suit the expected sea conditions. Structures may be offshore or onshore, or they may project out from the shore as in *groynes*, which are also used in river bank protection.



Wave interactions with structures are complex and non-linear. Extreme care must be taken not to exacerbate problems or create new ones, in particular ecological problems or shoreline erosion. Scale model testing and reliance on experience are essential. Other than this brief mention, for which we take no responsibility, this is an area of specialist practice outside of a first course in fluid mechanics.

Easy Derivation of River Wave Celerity

Consider a single wave of height H in stationary water of depth d . Superimpose the unknown celerity c to place the system in a moving reference frame. Then the fluid appears to approach a stationary wave with velocity c and passes under the wave at a velocity $V < c$ (reduced by continuity):



Continuity:

$$V A = \text{constant}$$

$$c dw = V (d + H)w \quad (w \text{ is the width of wave into the page})$$

$$\text{or} \quad V = cd/(d+H)$$

Balance of Pressure Force and Momentum:

The mean hydrostatic pressure occurs at the mean depth = depth/2. The hydrostatic force F is the mean pressure x the area. Force differences are balanced by changes in momentum $\rho Q \Delta V$:

$$F_c - F_t = \rho Q \Delta V$$

$$\text{or} \quad \rho g(d + H)^2/2 - \rho g d^2/2 = \rho c d (c - V)$$

$$g(d^2 + 2dH + H^2)/2 - g d^2/2 = c d (c - cd/(d+H))$$

$$g(2dH + H^2)/2 = c^2 d ((d + H - d)/(d+H))$$

$$g(2d + H)H/2 = c^2 d H (1/(d+H))$$

$$g(2d + H)/2 = c^2 d (1/(d+H)) \quad : \text{A classic process of } \textit{expand, incorporate and eliminate}.$$

For a wave train, the mean water depth y is in the middle of the wave's amplitude, not at its base, so a substitution $d = y - H/2$ can be made:

$$g(2(y - H/2) + H)/2 = c^2 y (1/((y - H/2) + H))$$

$$g y = c^2 y (1/(y + H/2))$$

Now assuming $y \gg H/2$, ie $y + H/2 \sim y$:

$$g y = c^2 y (1/y)$$

$$g y = c^2$$

$$\text{or} \quad c = (g y)^{1/2} \text{ as required}$$

The assumption $y \gg H/2$ is valid for natural waves, ie those forming as a result of energy exchanges due to the river's state of flow. This simple derivation does not follow the 2-D potentiometric method used by Airy (1857), but gives the same result.

Note: If we cannot assume $H/2 \ll y$ the relation becomes

$$c = (g(y + H/2))^{1/2} = (g(y + A))^{1/2}$$

Which is non-linear and less analytic than $c = (g y)^{1/2}$.

References:

- Chadwick, A and Morfett, J. 1993. *Hydraulics in Civil and Environmental Engineering* Second Edition (2nd Ed has a better section on wave theory). E & FN Spon. Dixon 627/C132h/1993
- Kay, M. 2008. *Practical Hydraulics*. Taylor and Francis, London and New York
- Kim, Y.C. (Ed) (2010). *Handbook of Coastal and Ocean Engineering*. World Scientific. Dixon 620.4162/H236