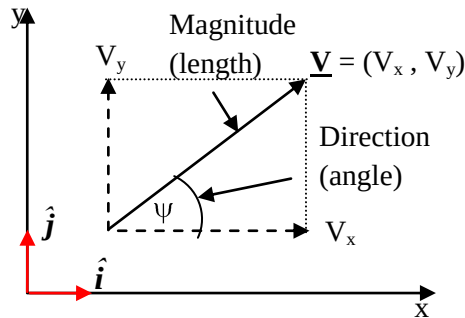


Vector Calculus: A Refresher for Fluid Mechanics

Fluid mechanics deals with *directional* quantities such as velocity and force, as well as pressure gradients and diffusive processes that depend on directional rates. For this reason a good understanding of vector algebra and calculus is essential for fluid mechanics.

Definition: A *vector* is a quantity with both magnitude and direction: eg velocity:



- eg a fluid particle
- an aircraft flight path

V_x and V_y are the components of $\underline{V}$. In 3 dimensions, V_z is in the vertical direction.

A common notation is to show $\underline{V}$ as a collection of components in x, y, z:

$$\underline{V} = (V_x, V_y, V_z)$$

Another common notation uses the *unit vectors* $\hat{i}$, $\hat{j}$ and $\hat{k}$ as vector multipliers of unit length in the x, y and z directions:

$$\underline{V} = V_x \hat{i} + V_y \hat{j} + V_z \hat{k}$$

If magnitude and angles are known, components can be determined, and vice versa:

$$V_x = |\underline{V}| \sin(\phi) \cos(\psi)$$

$$V_y = |\underline{V}| \sin(\phi) \sin(\psi)$$

$$V_z = |\underline{V}| \cos(\phi)$$

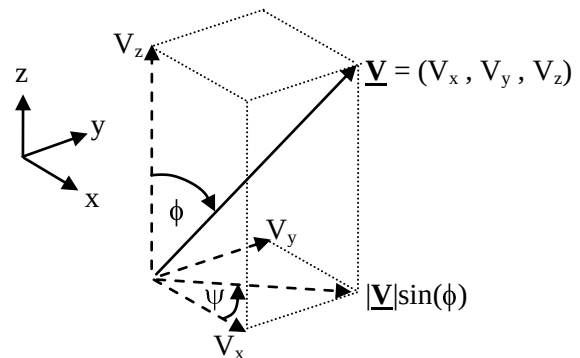
A vector can be described by its magnitude and angle:

$$|\underline{V}| = \sqrt{V_x^2 + V_y^2 + V_z^2} = \text{magnitude}$$

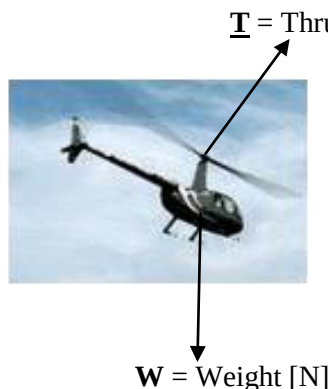
$$\psi = \tan^{-1}\left(\frac{V_y}{V_x}\right) = \text{angle in xy plane } \pm 180^\circ$$

In 3-D we need an angle to the vertical:

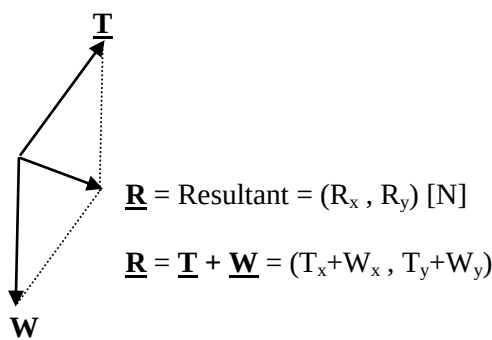
$$\phi = \cos^{-1}\left(\frac{V_z}{|\underline{V}|}\right) = \text{angle to z-axis}$$



Vector Addition: Vectors are added by summation of their components: eg Force summation:



Free Body Diagram



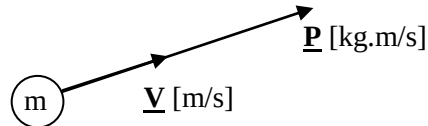
Force Diagram

Note: Vector summation is subject to the usual rules of addition, for instance, two vectors must have the same units before you can sum them (ie you can't add apples and oranges, or force and velocity). Vector subtraction works in the same way, ie for $\underline{A} = \underline{B} - \underline{C}$ we have $(A_x, A_y) = (B_x - C_x, B_y - C_y)$. The difference between two velocity vectors is a *relative velocity*.

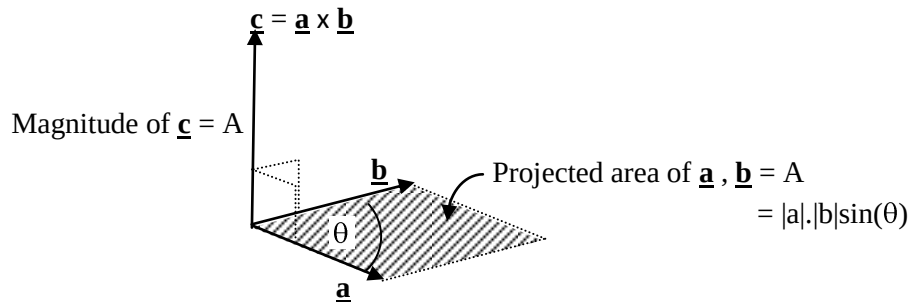
Vector Multiplication: Vectors can be multiplied in a number of ways, the three most useful (for fluid mechanics) being:

- a) Multiplication: a vector multiplied by a scalar (any number) gives another vector of the same direction but different magnitude: eg velocity and momentum, both vector quantities, are related by the scalar quantity *mass*:

$$\underline{\mathbf{P}} = m\underline{\mathbf{V}} = (P_x, P_y) = (mV_x, mV_y) = P_x\hat{\mathbf{i}} + P_y\hat{\mathbf{j}}$$



- b) Vector Cross-Product: a vector cross-multiplied by another vector gives a third vector at right angles (*normal*) to the first two and of magnitude equal to the area of the parallelogram formed by the first two:



$$\underline{\mathbf{c}} = \underline{\mathbf{a}} \times \underline{\mathbf{b}} = (c_x, c_y, c_z) = c_x\hat{\mathbf{i}} + c_y\hat{\mathbf{j}} + c_z\hat{\mathbf{k}}$$

$$\text{where } c_x = (a_y b_z - a_z b_y)$$

$$c_y = (a_z b_x - a_x b_z)$$

$$c_z = (a_x b_y - a_y b_x)$$

The magnitude of the cross-product is given by

$$|\underline{\mathbf{c}}| = |\underline{\mathbf{a}}| \cdot |\underline{\mathbf{b}}| \cdot \sin(\theta) = \text{Area}(\underline{\mathbf{a}}, \underline{\mathbf{b}})$$

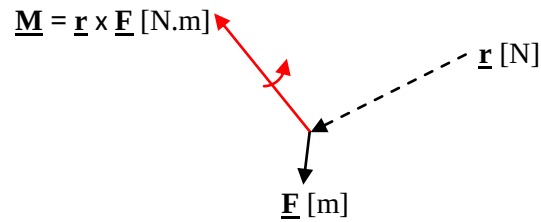
Where θ is the angle between $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$ and the $\cdot$ indicates ordinary scalar multiplication.

A common notation and useful reminder for the order of multiplication, addition and subtraction in a vector cross-product is:

$$\underline{\mathbf{c}} = \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{pmatrix} \begin{matrix} a_x \\ b_x \end{matrix}$$

That is arrange the two vectors in parallel lines, add the ↘ products and subtract the ↙ products.

While the cross-product can be interpreted as an expression of the projected area of two vectors it has many applications unrelated to area: for example the Torque or Moment exerted by a force around a point is equal to a cross-product of a leverage or position vector $\underline{r}$ and a force $\underline{F}$ at that point:

$$\underline{M} = \underline{r} \times \underline{F} \text{ [N.m]}$$




The direction of a cross-product vector and the resulting turning action of moments (indicated by the turned arrow) can be determined by the right hand rule: point your right hand towards the principal vector (eg the position vector) and curl your fingers towards the second vector (eg the force): the cross-product is now parallel to your thumb.

This means that a cross-product in the opposite order is in the opposite direction:

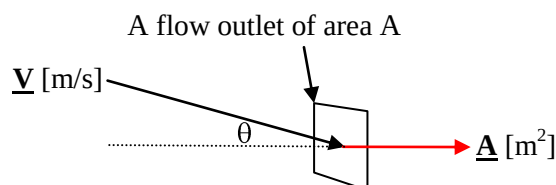
$$\underline{a} \times \underline{b} = - \underline{b} \times \underline{a}$$

Cross-products must be done in the right order, or the answer comes out backwards.

- c) Scalar or Dot-Product: the *parallel* components of two vectors are multiplied to produce a scalar equivalent to their combined effect in that direction. The dot product can be interpreted geometrically as the product of vector magnitudes and the cosine of the angle between them, or as a sum of their multiplied components:

$$\underline{a} \cdot \underline{b} = |\underline{a}| \cdot |\underline{b}| \cdot \cos(\theta) = a_x b_x + a_y b_y + a_z b_z : \text{a scalar}$$

Applications include net force in a given direction, or flowrate across a surface aligned in a given direction:



The flow velocity $\underline{V}$ is misaligned (by an angle θ) with the outlet area. What is the net flowrate Q in m^3/s across the outlet surface area?

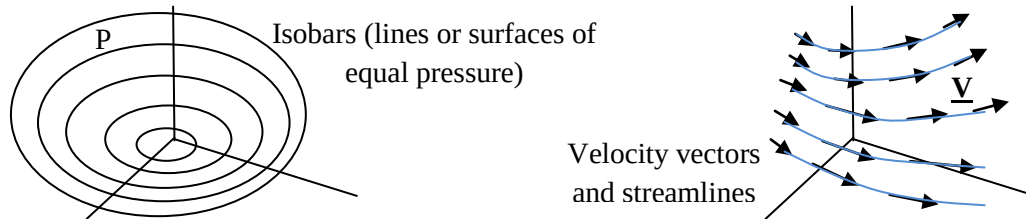
The outlet is aligned by the vector $\underline{A}$ normal to the surface and of magnitude A . The component of $\underline{V}$ parallel to $\underline{A}$ is $|\underline{V}| \cdot \cos(\theta)$. Thus the net flowrate across the surface is:

$$Q = |\underline{V}| \cdot |\underline{A}| \cdot \cos(\theta) [\text{m}^3/\text{s}] = \underline{V} \cdot \underline{A}$$

The dot product produces a scalar, ie a number with magnitude (positive or negative) but no direction. It follows that you can dot-multiply in any order and get the same answer, ie $\underline{V} \cdot \underline{A} = \underline{A} \cdot \underline{V}$

Vector and Scalar Fields: Quantities such as pressure, temperature and velocity vary with the geometry and dynamics of flow systems. They can be described mathematically as *fields* of space- and time-variable functions:

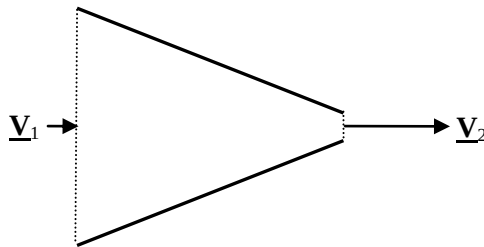
For example pressure is a scalar field (pressure has no direction) $P = P(x, y, z, t)$, while velocity is a vector field $\underline{V} = (V_x(x, y, z, t), V_y(x, y, z, t), V_z(x, y, z, t))$. The two types of field are very different:



Field Differentiation: Vector and scalar fields can be differentiated to find the principal rates of change in space and time dimensions.

A number of differential operations useful in fluid mechanics are set out below. The key to vector differentiation is to *perform the same operation on all of the components of each vector*. This can become complex and requires that you organise your components and operations in a methodical way, say for example three steps for the x, y and z directions with a fourth recombining step at the end.

Chain Rules: Often we can identify unknowns as combinations of known derivatives: for example the acceleration of a fluid particle through a steady, non-uniform flow can be determined by measuring its velocity and the rate at which velocity changes with distance:



Steady flow through a nozzle accelerates at a uniform rate determined by the shape of the nozzle. Each particle accelerates as it moves through this flow field. The acceleration is determined by how the velocity changes with distance, and how quickly each particle proceeds through that change as determined by its velocity:

$$\text{Thus if } V_x = V(x(t)), \text{ we have an acceleration } a_x = \frac{dV_x}{dt} = \boxed{\frac{dV_x}{dx} \cdot \frac{dx}{dt}} = \frac{dV_x}{dx} \cdot V_x$$

convective chain rule

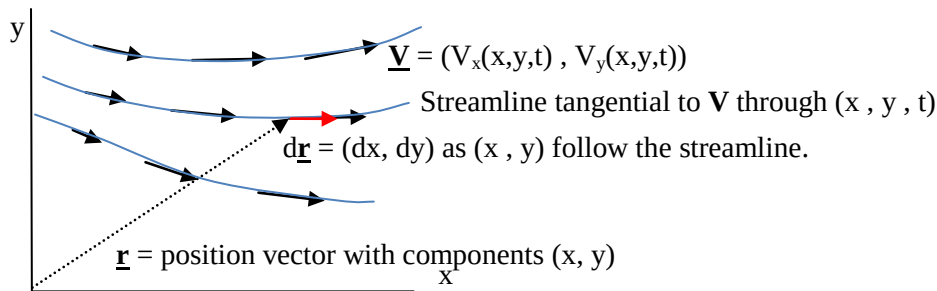
A number of useful chain rules are covered in maths courses. As a general rule, where a vector's components are variable functions of space and time, you need to differentiate with respect to all possible directions of change including time:

That is for $\underline{V} = (V_x, V_y) = (f_x(x,y,t), f_y(x,y,t))$: a vector function

$$\text{We have } \frac{d\underline{V}}{dt} = \left(\frac{\partial V_x}{\partial t} + \frac{\partial V_x}{\partial x} \frac{dx}{dt} + \frac{\partial V_x}{\partial y} \frac{dy}{dt}, \frac{\partial V_y}{\partial t} + \frac{\partial V_y}{\partial x} \frac{dx}{dt} + \frac{\partial V_y}{\partial y} \frac{dy}{dt} \right)$$

Where ∂ indicates partial differentiation with respect to the specified variable. The components of a vector are ordinary scalar functions and can be differentiated as per the general rules of calculus. The differentiated components are then components of another vector, in this case an acceleration vector.

Streamlines: A velocity vector field is a description of flow velocity at each point in space and time, for a given case or geometry within a defined boundary. At any instant, each flow element moves on a streamline that is tangential (parallel) to the flow at that point:



The streamline can be regarded as a function of $y = f(x)$ in which x and y change by dx and dy in a time interval dt .

That is

$$\underline{S} = d\underline{r} = (dx, dy)$$

$\underline{S}$ is parallel to $\underline{V}$, hence

$$\theta_{s,v} = 0 \text{ (angle between } \underline{S} \text{ and } \underline{V}\text{)}$$

or

$$\sin(\theta_{s,v}) = 0$$

equally

$$|\underline{V} \times \underline{S}| = |\underline{a}| \cdot |\underline{b}| \cdot \sin(\theta_{s,v}) = 0$$

Thus

$$|\underline{V} \times \underline{S}| = |\underline{V} \times (dx, dy)| = 0$$

The cross-product of a velocity and its associated streamline is always zero. This can be used to extract a function for the streamline which can then be plotted as a means of profiling the velocity field. Extending this to three dimensions x , y and z we have:

$$\underline{V} \times d\underline{r} = (0, 0, 0)$$

$$\text{where } 0 = (V_y dz - V_z dy)$$

$$0 = (V_z dx - V_x dz)$$

$$0 = (V_x dy - V_y dx) \text{ as for the definition of a cross-product (above)}$$

$$\text{hence } \frac{dy}{dz} = \frac{V_y}{V_z}; \frac{dz}{dx} = \frac{V_z}{V_x}; \frac{dy}{dx} = \frac{V_y}{V_x}$$

These differential equations can be integrated to give streamline functions in the y - z , x - z and x - y planes.

For 2-D problems, where $\underline{V} = (V_x, V_y)$ we have $\frac{dy}{dx} = \frac{V_y}{V_x}$ as the differential equation of a streamline.

Integrating dy/dx , the integration constant is determined by the initial values (x_0, y_0) . For a range of initial positions we get a range of integration constants, giving a family of curves (eg the three curves shown in the figure) defining streamlines through the vector field within a defined boundary.

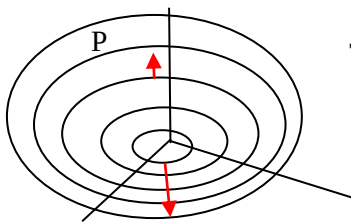
The Gradient or Del Operator:

The gradient operator is a vector indicating the direction and magnitude of maximum rate of increase of a scalar or vector field at any point. This useful vector operator arises in almost all vector descriptions of fluids:

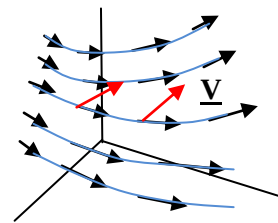
$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \text{ is the operator ; } f = f(x, y, z, t)$$

For example pressure as a scalar field P and velocity as a vector field $\mathbf{V}$ as before:



The gradient operator indicates the direction and magnitude of fastest increase in a scalar or vector field.



Divergence of a Vector Field:

Divergence is a scalar measure of the difference between flowrates entering and leaving a volume. This can be applied to the infinitesimal volume at a point to determine rates of compression and changes in density. With positive divergence, flowlines tend away from a point, while with negative divergence the flow lines tend to converge.

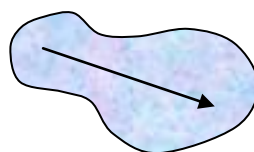
$$\text{Div}(\mathbf{V}) = \nabla \cdot \mathbf{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} : \text{a scalar value}$$

Incompressible flows have zero divergence everywhere, meaning that any divergent flow direction is compensated by a convergent flow from another direction. The mass continuity equation relates density changes to the divergence of a momentum field:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

For incompressible flows, density does not change, therefore the divergence of the vector field is zero.

You can visualise the zero-divergence concept by squeezing a water-filled balloon: any pressure inwards is compensated by an equal 'bulging' outwards of the balloon.



The situation is different, and to different degrees, for compressible flows. The balloon analogy becomes difficult as the flowrates required to induce compression effects are much higher.

Gauss' Divergence Theorem:

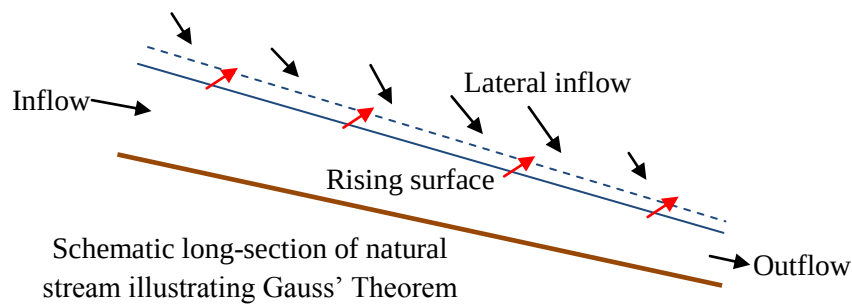
The total flux of a vector field across a closed surface equals the total divergence through the enclosed volume:

$$\oint \underline{V} \cdot \underline{\hat{n}} dA = \iiint \nabla \cdot \underline{V} dV$$

Where $\underline{V} \cdot \underline{\hat{n}}$ is a vector flux $\underline{V}$ over a surface A with normal vector $\underline{\hat{n}}$ and $\nabla \cdot \underline{V}$ is the divergence of $\underline{V}$ within the volume $\forall$.

The $\oint$ surface and $\iiint$ volume integrals are usually eliminated in the application of this theorem (so you don't necessarily need to know how to do them!) but are essential for this definition.

Again, this complex relationship can be visualised by the incompressible behaviour of water: a natural stream rises and falls as flows enter and exit at upstream and downstream ends or along the length of the stream (the surfaces of the volume). Since the divergence of water is everywhere zero (incompressible) the flowrates on all surfaces entering and leaving the volume must be equal, including those flows that produce the rising and falling free surface. Compressible flows have a wider and more complex range of behaviour around this general idea.



Gauss' theorem is a very powerful general relationship. It provides a conversion from surface to volume integrals which leads to easier or tractable solutions in many complex problems. As such it is the basis of most dynamic methods in fluid mechanics.

Curl of a Vector Field:

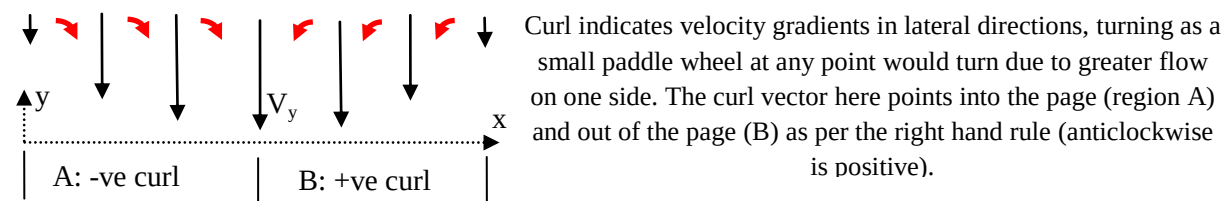
Velocity gradients (rates at which velocity changes from place to place) tend to create rotations in flow. These are observable at a fine macroscopic scale, more easily using fine-particle tracers such as dust or tea leaves. Turbulence tends to amplify rotations into large-scale eddies and vortices.

Conveniently, there is a rotational vector to describe it:

$$\text{Curl}(\underline{V}) = \nabla \times \underline{V}$$

$$= \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}, \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}, \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right)$$

$$= \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$



In a two dimensional flow $\underline{V} = (V_x(x,y,t), V_y(x,y,t))$ the curl vector reduces to:

$$\nabla \times \underline{V} = \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \hat{k}$$

Curl is sometimes referred to as *vorticity*, ξ . Vorticity arises due to viscous effects that create velocity distributions through the transfer of momentum. Flows with zero vorticity are irrotational, usually meaning they are idealised models with zero friction.

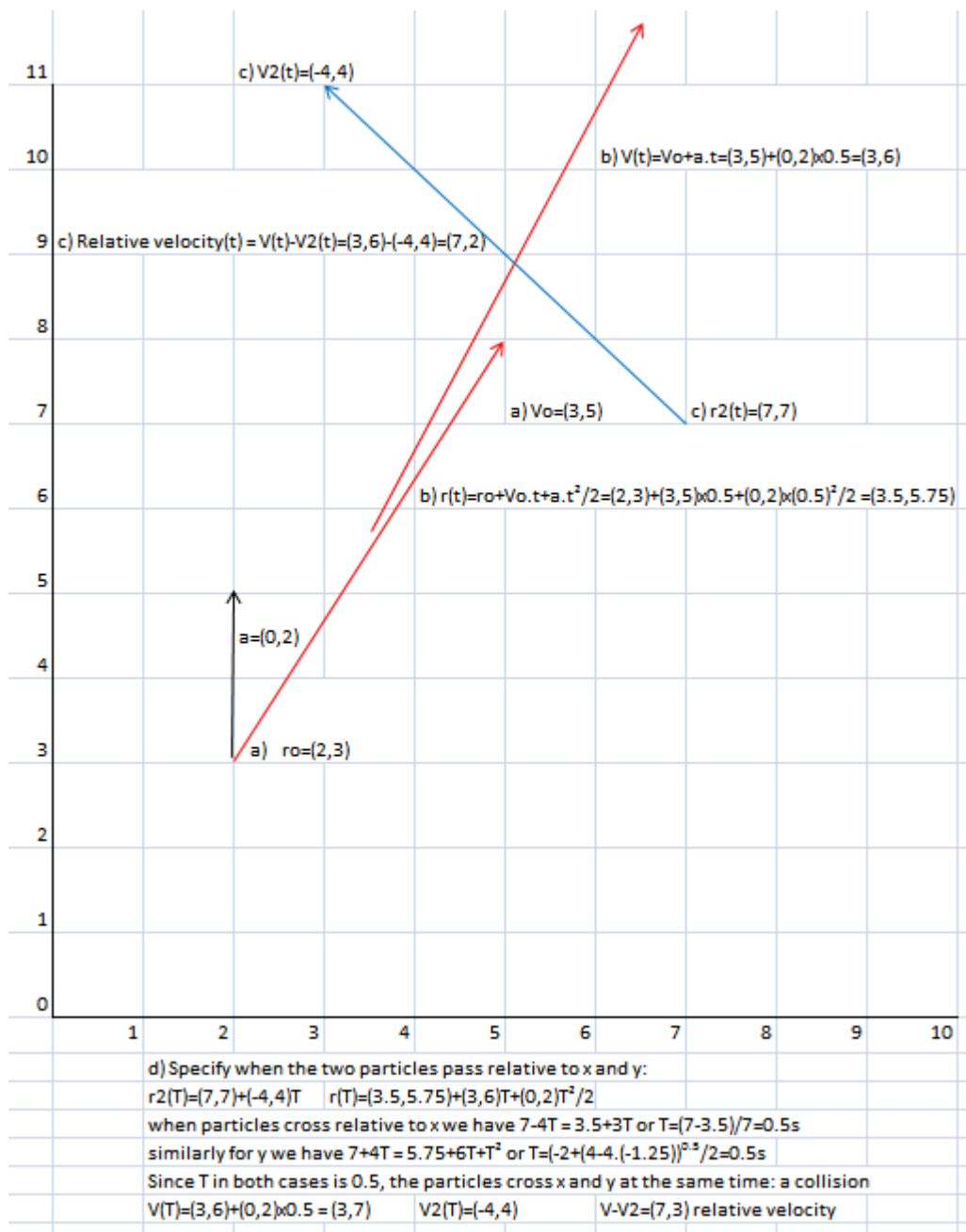
Exercises

- 1) Use graph paper to plot and formulate vectors:
 - a. Sketch the velocity vector $\mathbf{V} = (3, 5)$ m/s for a particle at the position vector $\mathbf{r} = (2, 3)$ m.
 - b. If the acceleration on the particle in a. is constant $\mathbf{a} = (0, 2)$ m/s² locate the position and velocity of the particle after 0.5 seconds.
 - c. At that moment a second particle is at $\mathbf{r} = (7, 7)$ moving at $\mathbf{V}_2 = (-4, 4)$. What is the relative velocity between the two particles at that moment?
 - d. Are these particles on a collision course? What is their relative velocity at nearest approach?
- 2) An air-gun pellet with mass 0.001 kg is fired at a 20° angle to the horizontal at 60 m/s. Assuming gravity and wind resistance have a negligible effect on a pellet at short range:
 - a. Find the components of the velocity and momentum vectors at short range.
 - b. The pellet strikes the centre of a 10 cm high vertical target with an inelastic collision, bouncing off with effectively zero horizontal speed, but with its vertical speed unchanged. How much momentum is transferred to the target as an impulse?
 - c. The target, actually a tin duck, is designed to fall over on an impulse moment of at least 3×10^{-3} kg m²/s. Is the shooting gallery rigged, or only rigged for little kids? Should small people aim for the duck's head or body?
 - d. The target's head is located by a vector $\mathbf{h} = (0, 0.1)$, while its body is $\mathbf{b} = (0, 0.05)$. Confirm your conclusions in c. by the moment-impulse cross-product $\mathbf{r} \times \mathbf{P}$ where $\mathbf{P}$ is a momentum vector and $\mathbf{r}$ is a leverage. Use the components of the vectors.
- 3) A certain hill in the centre of town is a uniform slope that is steeper to the east than it is to the north. The hill's altitude z relative to the nearby post office is approximated by the equation
$$z = 0.05x + 0.03y \quad \text{m}$$
where x and y are distances (in metres) east and north of the post office.
 - a. When rain falls on the hillside, in which direction does it flow down the hill?
 - b. If the handbrake on your car failed, which ways could you face it to park safely?
 - c. A kid in a go-kart takes off in a due south direction. What is his initial acceleration?
- 4) The flow velocity in a smooth contraction is approximated by:
$$\mathbf{V} = \left(1, -\frac{y}{2}\right) \quad \text{m/s}$$
 - a. Plot streamlines as $y = f(x)$ from the initial positions $(1, 1)$ and $(1, -1)$.
 - b. What is the convective acceleration of the flow at $(2, 0)$? Does this seem plausible for a contraction?
 - c. Is this flow compressible? What modification could be made to the horizontal velocity to model an incompressible flow? (hint: use the divergence of the velocity field). What is the convective acceleration in the incompressible model?
 - d. What is the vorticity of the compressible and incompressible flow models? What does this tell you about the viscosity of the flow in either case?

Worked solutions

1) Use graph paper to plot and formulate vectors:

- Sketch the velocity vector $\mathbf{V} = (3, 5)$ m/s for a particle at the position vector $\mathbf{r} = (2, 3)$ m.
- If the acceleration on the particle in a. is constant $\mathbf{a} = (0, 2)$ m/s² locate the position and velocity of the particle after 0.5 seconds.
- At that moment a second particle is at $\mathbf{r} = (7, 7)$ moving at $\mathbf{V}_2 = (-4, 4)$. What is the relative velocity between the two particles at that moment?
- Are these particles on a collision course? What is their relative velocity at nearest approach?



- 2) An air-gun pellet with mass 0.001 kg is fired at a 20° angle to the horizontal at 60 m/s. Assuming gravity and wind resistance have a negligible effect on a pellet at short range:
- Find the components of the velocity and momentum vectors at sort range.
 - The pellet strikes the centre of a 10 cm high vertical target with an inelastic collision, bouncing off with effectively zero horizontal speed, but with its vertical speed unchanged. How much momentum is transferred to the target as an impulse?
 - The target, actually a tin duck, is designed to fall over on an impulse moment of at least $3 \times 10^{-3} \text{ kg m}^2/\text{s}$. Is the shooting gallery rigged, or only rigged for little kids? Should people aim for the duck's head or body?
 - The target's head is located by a vector $\mathbf{h} = (0, 0.1)$, while its body is $\mathbf{b} = (0, 0.05)$. Confirm your conclusions in c. by the moment-impulse cross-product $\mathbf{r} \times \mathbf{P}$ where $\mathbf{P}$ is a momentum vector and $\mathbf{r}$ is a leverage. Use the components of the vectors.

a) $\underline{\mathbf{V}} = (60\cos(20), 60\sin(20)) = (56.4, 20.5) \text{ m/s}$

$\underline{\mathbf{P}} = m\underline{\mathbf{V}} = 0.001(56.4, 20.5) = (0.0564, 0.0205) \text{ kg m/s}$

- b) The pellet elastically collides with the target over some short time interval, creating a brief period t of force $\underline{\mathbf{F}}$ against the target, known as an impulse, $t\underline{\mathbf{F}}$. The total impulse delivered is equal to the momentum change, $\Delta\underline{\mathbf{P}} = (0.0564, 0.0205) - (0, 0.0205) = (0.0564, 0) \text{ kg m/s}$

This horizontal component of the momentum is thus transferred to the target as an impulse.

- c) If the pellet hits the target at its centre, ie 5 cm from the hinge, the impulse moment is the impulse acting as a turning moment about that leverage distance:

ie $M = 0.05(0.0564) = 2.82 \times 10^{-3} \text{ kg m}^2/\text{s}$

This is less than the required impulse moment of $3 \times 10^{-3} \text{ kg m}^2/\text{s}$. The game is rigged!

Small people are likely to have a large initial shooting angle. Adults would be more able to hit the target dead on, thus the whole component of their pellet's momentum is horizontal:

$\underline{\mathbf{P}} = 0.001(60, 0) = (0.06, 0)$

And $M = 0.05(0.06) = 3 \times 10^{-3} \text{ kg m}^2/\text{s}$.

Just enough. It would be better to aim at the head, which gives a larger leverage for the impulse moment:

$M = 0.1(0.06) = 6 \times 10^{-3} \text{ kg m}^2/\text{s}$.

- d) To confirm this conclusion, use the impulse moment product $\underline{\mathbf{r}} \times \underline{\mathbf{P}}$ for head and body:

Body: $\underline{\mathbf{r}} = (0, 0.05)$ $\underline{\mathbf{P}} = (0.0564, 0.0205)$:

$\underline{\mathbf{r}} \times \underline{\mathbf{P}} = (0 \times 0.0205 - 0.05 \times 0.0564) = (0, 0, -2.82 \times 10^{-3}) \text{ kg m}^2/\text{s}$

This is an impulse moment vector with magnitude $2.82 \times 10^{-3} \text{ kg m}^2/\text{s}$. Not enough to knock over the duck.

Head: $\underline{\mathbf{r}} = (0, 0.1)$:

$\underline{\mathbf{r}} \times \underline{\mathbf{P}} = (0 \times 0.0205 - 0.1 \times (0.0564)) = (0, 0, -5.64 \times 10^{-3}) \text{ kg m}^2/\text{s}$

This is an impulse moment vector with magnitude $5.64 \times 10^{-3} \text{ kg m}^2/\text{s}$. Knocks the duck cold!

- 3) A certain hill in the centre of town is a uniform slope that is steeper to the east than it is to the north. The hill's altitude z relative to the nearby post office is approximated by the equation

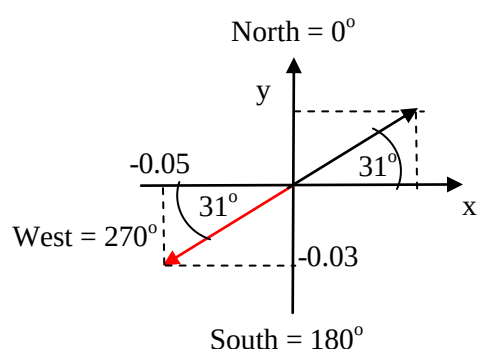
$$z = 0.05x + 0.03y \text{ m}$$

where x and y are distances (in metres) east and north of the post office.

- When rain falls on the hillside, in which direction does it flow down the hill?
 - If the handbrake on your car failed, which ways could you face it to park safely?
 - A kid in a go-kart takes off in a due south direction. What is his initial acceleration?
- a) We need the gradient function of the hillside to determine the direction of steepest increase, or uphill. Downhill (the way the rainwater flows) will be 180° from that.

$$\nabla z = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right) = (0.05, 0.03) \text{ m/m, a slope vector}$$

The downhill vector is thus $(-0.05, -0.03)$, a vector in the opposite direction to ∇z . The angle of the vector is $\tan^{-1}(-0.03/-0.05) = 31^\circ$ to due west: $270 - 31 = 239^\circ$ or SW



Angular calculations can be hard for two reasons:

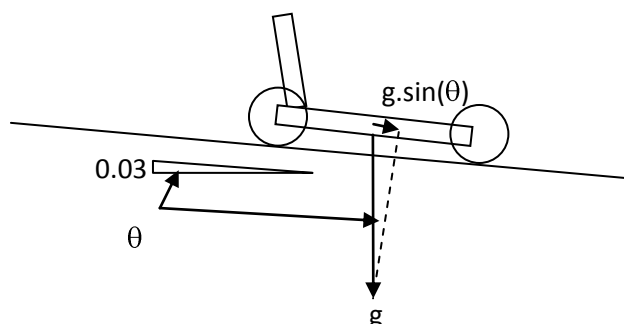
1: The arctan formula cannot distinguish between a vector and its negative, thus the two vectors shown here produce the same result: 31° . You need to draw a diagram to determine if the angle is defined relative to the x or the $-x$ axis.

2: Mathematics and geography use different definitions: Mathematicians define an angle of zero (0°) as parallel to the x -axis, positive anticlockwise. Geographers use the y -axis as the reference for zero (North = 0°), with positive clockwise (East = 90°). Compass points imply the geographic convention.

- You can park a loose car in two directions on a slope, both pointing along the line of constant height. Either direction is at 90° to ∇z , ie $239 + 90 = 329^\circ$ NW or $239 - 90 = 149^\circ$ SE
- Due south is a vector $\underline{u} = (0, -1)$. The slope in that direction is -0.03 m/m , ie the slope of z in the $-y$ direction. In general a slope in a direction is a dot product:

$$\nabla_u = \underline{u} \cdot \nabla z = (0, -1) \cdot (0.05, 0.03) = 0 \times 0.05 + -1 \times 0.03 = -0.03 \text{ m/m}$$

The slope angle is $\theta = \tan^{-1}(-0.03) = -1.7^\circ$. The acceleration in the due south direction is then $a = g \cdot \sin(1.7^\circ) = 0.29 \text{ m/s}^2$



- 4) The flow velocity in a smooth contraction is approximated by:

$$\mathbf{V} = \left(1, -\frac{y}{2} \right) \text{ m/s}$$

- Plot streamlines as $y = f(x)$ from the initial positions $(1, 1)$ and $(1, -1)$.
 - What is the convective acceleration of the flow at $(2, 0)$? Does this seem plausible for a contraction?
 - Is this flow compressible? What modification could be made to the horizontal velocity to model an incompressible flow? (hint: use the divergence of the velocity field). What is the convective acceleration in the incompressible model?
 - What is the vorticity of the compressible and incompressible flow models? What does this tell you about the viscosity of the flow in either case?
- a) The differential equation of a streamline in 2 dimensions is

$$\frac{dy}{dx} = \frac{V_y}{V_x} = -\frac{y}{2}$$

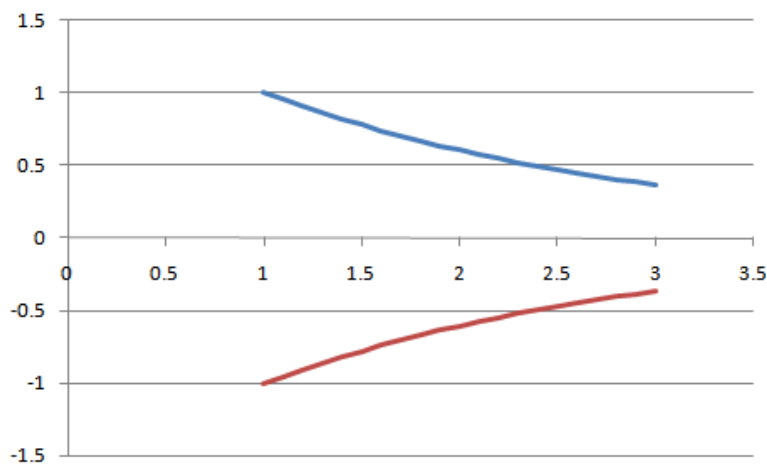
So
$$\frac{dy}{y} = -\frac{dx}{2}$$

$$\ln(y) = -\frac{x}{2} + c$$

$$y = ae^{-\frac{x}{2}}$$

For $(1, 1)$, $a = 1.65$

For $(1, -1)$, $a = -1.65$



- b) The convective acceleration is given by

$$\frac{d\mathbf{V}}{dt} = \left(\frac{\partial V_x}{\partial t} + \frac{\partial V_x}{\partial x} V_x + \frac{\partial V_x}{\partial y} V_y, \frac{\partial V_y}{\partial t} + \frac{\partial V_y}{\partial x} V_x + \frac{\partial V_y}{\partial y} V_y \right)$$

The flow is steady (not a function of time) so $\frac{\partial V}{\partial t} = 0$. In fact all terms other than $\frac{\partial V_y}{\partial y}$ are 0:

$$\frac{d\mathbf{V}}{dt} = \left(0, \frac{\partial V_y}{\partial y} V_y \right) = \left(0, \frac{y}{4} \right)$$

At (2, 0) this is zero. There is no horizontal acceleration, which seems inconsistent for a contraction.

- c) The convective acceleration is partly determined by the divergence of the flow. The flow may be compressible, with an artificial horizontal velocity that results in compression:

$$\text{Div}(\mathbf{V}) = \nabla \cdot \mathbf{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = -\frac{1}{2}$$

The fact that this is not zero indicates that the flow is compressible. We can reduce it to zero by modifying V_x :

$$\text{Div}(\mathbf{V}) = 0 \text{ when}$$

$$\frac{dV_x}{dx} = \frac{1}{2} \quad ; \quad V_x = \frac{x}{2}$$

Thus $\mathbf{V} = (x/2, -y/2)$ is an incompressible contraction. The convective acceleration is then $\frac{d\mathbf{V}}{dt} = \left(\frac{\partial V_x}{\partial y} V_x, \frac{\partial V_y}{\partial y} V_y \right) = \left(\frac{x}{4}, \frac{y}{4} \right)$

This provides a positive horizontal acceleration of $x/4$.

- d) The vorticity of a 2-dimensional flow is given by the curl:

$$\nabla \times \mathbf{V} = \left(0, 0, \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right)$$

= (0, 0, 0) for both incompressible and compressible models: The vorticity is zero, an ideal fluid condition that indicates the viscosity is modelled as zero.