
Supplementary Tutorial – Isentropic One-Dimensional Flow

This brief tutorial is intended to give you the tools for performing simple high speed gas flow calculations that incorporate the advanced thermodynamic theory. No derivations of the formulae are given. The emphasis is on practical application in a limited class of problems, namely adiabatic gas flows in short conduits. This includes gas leaks and simple gas nozzles. Examples demonstrating the use of the theory in these problems are given at the end. An advanced text is also provided in the Complete Lectures section online.

Definition: A gas flow which is insulated against heat exchange with the external environment is known as an *adiabatic* flow. If points of interest in the flow can be regarded as approximately frictionless, ie processes that take place over short distances and with minimal joint or process losses, the flow can be approximated as *isentropic* (same-entropy) with the following simplified one-dimensional theory:

Stagnation

A gas flow has initial conditions of absolute pressure P , density ρ , absolute or Kelvin-scale temperature T and velocity V . If the flow is brought to rest (ie decelerated to zero velocity) it is said to be *stagnated*. Flow conditions stagnate by absorbing kinetic energy, with resulting changes in P , T , and ρ . The resulting conditions are known as the *stagnation properties* of the flow:

$$T_o = T + \frac{V^2}{2C_p} \quad \text{Stagnation temperature (K)}$$

$$P_o = P \left(\frac{T_o}{T} \right)^{\frac{\gamma}{\gamma-1}} \quad \text{Stagnation Pressure (Pa)}$$

$$\rho_o = \rho \left(\frac{T_o}{T} \right)^{\frac{1}{\gamma-1}} \quad \text{Stagnation Density (kg/m}^3\text{)}$$

C_p is the specific heat at constant pressure, ie the amount of energy required to raise a 1-kg mass of the gas by 1 degree Kelvin at constant pressure. For air, $C_p = 1,005 \text{ J/kgK}$ (look up for other gases)

γ is the ratio of specific heats, ie the ratio of C_p to C_v , the specific heat at constant volume. For air and other diatomic gases, $\gamma \sim 1.4$. (In some texts the over-used symbol k is used for γ).

If the gas is initially at rest, ie $V = 0$ as for a gas in a tank or calm atmosphere, then the initial conditions are the stagnation properties:

$$\text{For } V = 0 \quad : \quad T_o = T \quad P_o = P \quad \rho_o = \rho$$

Speed of Sound

For ideal gases,

$$c = \sqrt{\gamma R T} \quad \text{Speed of sound at a given temperature (m/s)}$$

R is the specific gas constant. For air, $R = 287 \text{ J/kgK}$

R is related to the specific heat and heat ratio C_p and γ as $R/C_p = \frac{\gamma - 1}{\gamma}$

eg at room temperature, $T = 300 \text{ K}$, $c = (1.4(287)300)^{1/2} = \underline{347.2 \text{ m/s}}$

Mach Number

The Mach Number Ma is a dimensionless ratio of the flow velocity to the sound speed. Ma is essential to the isentropic 1-D theory:

$$Ma = V/c$$

$Ma < 1$: Subsonic

$Ma = 1$: Sonic or critical

$Ma > 1$: Supersonic

Sonic Stagnation Properties

Given $T_o = T + \frac{V^2}{2C_p}$ Stagnation temperature (K)

Substitution of the sound speed formula yields: $V^2 = \gamma RT Ma^2$ with $R/C_p = \frac{\gamma - 1}{\gamma}$

$$\frac{T_o}{T} = 1 + \left(\frac{\gamma - 1}{2} \right) Ma^2 \quad \text{Stagnation Temperature Ratio}$$

$$\frac{P_o}{P} = \left(\frac{T_o}{T} \right)^{\frac{\gamma}{\gamma - 1}} = \left(1 + \left(\frac{\gamma - 1}{2} \right) Ma^2 \right)^{\frac{\gamma}{\gamma - 1}} \quad \text{Stagnation Pressure Ratio}$$

$$\frac{\rho_o}{\rho} = \left(\frac{T_o}{T} \right)^{\frac{1}{\gamma - 1}} = \left(1 + \left(\frac{\gamma - 1}{2} \right) Ma^2 \right)^{\frac{1}{\gamma - 1}} \quad \text{Stagnation Density Ratio}$$

$$V = \sqrt{2C_p(T_o - T)} = \sqrt{2C_p T \left(\frac{\gamma - 1}{2} \right) Ma^2} \quad \text{Flow velocity m/s}$$

In isentropic flow these equations can be used to determine:

- Stagnation properties from partial information on initial conditions T, P, ρ, V
- Final conditions given partial information on stagnation and final conditions T_f, P_f, ρ_f, V_f

a) eg given initial V and T , $Ma = V/\sqrt{\gamma RT}$ and all T_o, P_o, ρ_o can be solved

b) eg given stagnation pressure and temperature P_o and T_o and final pressure (or *back pressure*) P_f , the final temperature and other variables can be found:

$$T_f = T_o \left(\frac{P_f}{P_o} \right)^{\frac{\gamma - 1}{\gamma}} \quad \text{Final Temperature (K)}$$

$$V_f = \sqrt{2C_p(T_o - T_f)} \quad \text{Final Velocity (m/s)}$$

$$\rho_f = \rho_o \left(\frac{T_f}{T_o} \right)^{\frac{1}{\gamma - 1}} \quad \text{Final Density (kg/m}^3\text{)}$$

Mass Flowrate and Flow Choking

In steady compressible gas flow the mass flowrate is constant:

$$\dot{m} = \rho VA \quad \text{Mass Flowrate (kg/s)}$$

The mass flowrate is controlled by the stagnation properties of the flow and exit conditions such as flow area and back pressure:

$$\dot{m} = \frac{AMaP_o \sqrt{\gamma/RT_o}}{[1 + (\gamma - 1)Ma^2 / 2]^{(\gamma+1)/2(\gamma-1)}} \quad \text{Controlled Mass Flowrate (kg/s)}$$

Where Ma is the Mach number at an orifice of area A, found by consideration of stagnation and final flow properties. This formula can be differentiated with velocity, revealing a maximum flowrate that occurs when Ma = 1.

$$\dot{m}_{max} = AP_o \sqrt{\frac{\gamma}{RT_o}} \left(\frac{2}{\gamma + 1} \right)^{\frac{\gamma+1}{2(\gamma-1)}}$$

The maximum flowrate occurs when the back pressure is less than a critical pressure determined below, such that the flow is not constrained by pressure. Stagnation and final conditions may be varied to increase flow velocity (eg by lowering the back pressure still further), but this cannot increase flowrate above the maximum: the flow is subsequently said to be *choked*.

Sonic (Critical) Conditions

$$\text{Given } \frac{T_o}{T} = 1 + \left(\frac{\gamma - 1}{2} \right) Ma^2$$

For Ma = 1 we obtain the *critical conditions*:

$$\frac{T^*}{T_o} = \frac{2}{\gamma + 1} \quad \text{Critical Temperature Ratio}$$

$$\frac{P^*}{P_o} = \left(\frac{2}{\gamma + 1} \right)^{\frac{\gamma}{\gamma - 1}} \quad \text{Critical Pressure Ratio}$$

$$\frac{\rho^*}{\rho_o} = \left(\frac{2}{\gamma + 1} \right)^{\frac{1}{\gamma - 1}} \quad \text{Critical Density Ratio}$$

$$V^* = c^* = \sqrt{\gamma RT^*} \quad \text{Sonic Velocity}$$

Conditions at a throat cannot be supersonic: If the back pressure is reduced to less than P*, the conditions at a throat are not effected and P_{throat} = P*, V_{throat} = V*, $\dot{m} = \dot{m}_{max}$.

More Information?

In some problems the *critical area ratio* or *critical Mach number ratio* may be useful:

$$\frac{A^*}{A} = Ma \left[\left(\frac{2}{\gamma + 1} \right) \left(1 + \frac{\gamma - 1}{2} Ma^2 \right) \right]^{\frac{2(\gamma - 1)}{\gamma + 1}} \quad \text{Critical Area Ratio}$$

$$\frac{Ma^*}{Ma} = \sqrt{\frac{\gamma + 1}{2 + (\gamma - 1) Ma^2}} \quad \text{Critical Mach Number Ratio}$$

A* is the area of a throat that will produce sonic flow given an upstream area A flowing at $Ma < 1$

Ma* is the ratio of the local velocity to the speed of sound at a throat where critical conditions occur:

$$Ma^* = V/c^*$$

Energy Conservation

The stagnation properties provide a compressible analogue of Bernoulli's equation:

$$(q_{in} - q_{out}) + (w_{in} - w_{out}) + g(z_1 - z_2) + C_p(T_{o1} - T_{o2}) = 0 \quad \text{Energy Conservation (J/kg)}$$

q is heat per mass by radiation, conduction, convection etc into or out of a flow. w is work per mass by pumps (w_{in}) or turbines (w_{out}). Z is elevation. The use of stagnation temperatures T_{o1} and T_{o2} simplifies the calculation by including kinetic (V) and elastic (P) energies.

All terms have the units of energy per unit mass, which is equivalent to pressure per density P/ρ or power per mass flow rate $P/\rho Q$.

More on the Gas Parameters C_p , C_v and R ...

You only need to know γ and R in order to work out the parameters C_p and C_v :

$$C_p/C_v = \gamma$$

$$\text{And } C_p - C_v = R$$

$$\text{Thus } \gamma C_v - C_v = C_v (\gamma - 1) = R$$

$$C_v = R / (\gamma - 1)$$

$$\text{and } C_p = R + R / (\gamma - 1) = \gamma R / (\gamma - 1)$$

Examples

Air in an automobile tyre is at 220 kPa and 25 °C. The atmospheric pressure at this altitude is 94 kPa. If a 0.4 mm hole develops in the tyre what will be the initial mass and volume flowrate of the leak?

We need to work out the critical pressure at which sonic conditions occur. If the back pressure (atmospheric) is less than the critical pressure then the hole is not constrained by pressure, so exit conditions are sonic and the mass flowrate is the maximum, which is a relatively simple calculation.

If on the other hand the back pressure is more than critical then the flowrate is constrained to less than the maximum. This may require the calculation of a Mach number to find the flowrate.

Since the initial velocity is zero, the absolute tyre pressure is the stagnation pressure:

$$\begin{aligned}P_o &= 220 + 94 = 314 \text{ kPa} \\P^* &= P_o(2/(\gamma+1))^{\gamma/(\gamma-1)} \\&= 314 \times 10^3 (2/2.4)^{(1.4/0.4)} = 165.9 \text{ kPa}\end{aligned}$$

The back pressure is less than this critical value, so the hole is unconstrained by back pressure and critical conditions occur. The mass flowrate is thus given by the maximum value:

$$\dot{m} = \dot{m}_{\max} = A P_o \sqrt{(\gamma/R T_o) (2/(\gamma+1))^{\gamma/(\gamma-1)} ((\gamma+1)/(2(\gamma-1)))}$$

$$\text{where } A = \pi D^2/4 = \pi(0.0004)^2/4 = 125.7 \times 10^{-9} \text{ m}^2$$

$$\text{and } T_o = (273 + 25) = 298 \text{ K}$$

$$\text{so } \dot{m} = 125.7 \times 10^{-9} (314 \times 10^3) \sqrt{(1.4/(287(298))) (2/2.4)^{(2.4/0.8)} (2.4/0.8)} = 92.4 \times 10^{-6} \text{ kg/s}$$

or 92.4 mg/s. The density in the tyre is $\rho = P/RT = 314 \times 10^3/(287(298)) = 3.671 \text{ kg/m}^3$, and the volume flowrate is $Q = \dot{m}/\rho = 92.4 \times 10^{-6}/3.671 = 25.2 \times 10^{-6} = 25.2 \text{ ml/s}$

Air at 1 MPa absolute pressure and 600 °C enters a converging nozzle at a velocity of 150 m/s. Determine the mass flow rate for a throat area of 50 cm² and a back pressure of 0.7 MPa.

Although this is clearly not a common or domestic situation the same solution procedure applies. In this case the initial velocity is not zero, so we need to work out the stagnation properties:

$$\begin{aligned}T &= 600 + 273 = 873 \text{ K} \\T_o &= T + V^2/2C_p = 873 + 150^2/(2 \times 1005) = 884.2 \text{ K} \\P_o &= (T_o/T)^{\gamma/(\gamma-1)} = (884.2/873)^{(1.4/0.4)} = 1.046 \text{ MPa} \\P^* &= P_o(2/(\gamma+1))^{\gamma/(\gamma-1)} = 1.046(2/2.4)^{(1.4/0.4)} = 0.5526 \text{ MPa}\end{aligned}$$

The back pressure is greater than the critical pressure, so the throat is constrained by back pressure and subsonic conditions occur at the throat. We need to determine the velocity to solve for the mass flow rate:

The pressure at the throat is the back pressure P_b . The temperature at the throat can be determined from the stagnation pressure ratio:

$$T_b = T_o(P_b/P_o)^{(\gamma-1)/\gamma} = 884.2(0.7/1.046)^{0.4/1.4} = 788.3 \text{ K}$$

The throat velocity and celerity are then

$$V_b = \sqrt{2C_p(T_o - T_b)} = \sqrt{2 \times 1005(884.2 - 788.3)} = 439.0 \text{ m/s}$$

$$c_b = \sqrt{\gamma R T_b} = \sqrt{1.4(287)788.3} = 562.8 \text{ m/s}$$

The throat Mach number works out as $Ma_b = 439.0/562.8 = 0.78$

So the mass flow rate is

$$\dot{m} = \frac{A Ma P_o \sqrt{\gamma / R T_o}}{[1 + (\gamma - 1) Ma^2 / 2]^{(\gamma + 1)/(2(\gamma - 1))}} = \frac{50 \times 10^{-4} \times 0.78 \times 1.046 \times 10^6 \sqrt{1.4 / 287 \times 884.2}}{[1 + (0.4)0.78^2 / 2]^{(2.4)/(2(0.4))}}$$

$$= 6.79 \text{ kg/s}$$

Low pressure gas escape: Methane ($\gamma = 1.32$, $C_p = 2260 \text{ J/kgK}$) at 22 C and 37 kPa gauge pressure in a tank is fed by a smooth valve into a second tank of methane at atmospheric pressure. What is the flow velocity and temperature change?

From Sonic Stagnation Properties,

$$T_f = T_o \left(\frac{P_f}{P_o} \right)^{\frac{\gamma - 1}{\gamma}} \quad \text{Final Temperature (K)}$$

$$V_f = \sqrt{2C_p(T_o - T_f)} \quad \text{Final Velocity (m/s)}$$

$$\rho_f = \rho_o \left(\frac{T_f}{T_o} \right)^{\frac{1}{\gamma - 1}} \quad \text{Final Density (kg/m}^3\text{)}$$

$$T_f = 295(138.3/101.3)^{0.32/1.32} = 273 \text{ K}$$

ie the temperature drops by about 22 degrees (coincidence)

$$V_f = (2(2260(22)))^{0.5} = 315 \text{ m/s}$$

This is subsonic and broadly consistent with a Bernoulli analysis using the closed Pressure form of the equation $P + \rho V^2/2 + \rho g z$ and the gas constitutive law $P/\rho^\gamma = \text{constant}$.