

Supplementary Lecture 3: Boundary Layer Theory

Boundary layer theory had a profound effect on the development of 2- and 3-dimensional fluid mechanics in the 19th century. Its use now is limited, since most of these applications are routinely carried out by direct computation of the Navier-Stokes equations. The following lecture is provided for completeness, and for rare hand-calculation of surface drag on vehicles and boat hulls.

Boundary Layers

The effect of viscosity in the uniform flow region is small, approximately as given by Reynolds:

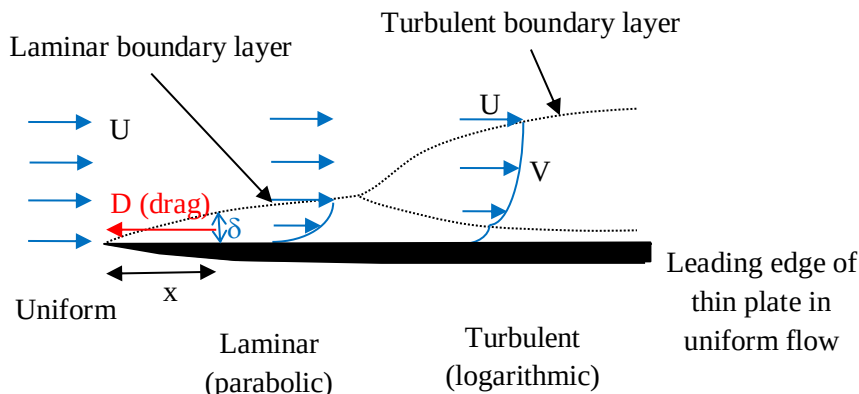
$$\tau_l = \mu \frac{dV}{dy} = \mu \frac{u'}{v' dt} = \mu \frac{u'}{\ell}$$

μ is a small constant, and the fluctuation velocity is usually a minor fraction of the flow velocity. The uniform flow region can thus be treated as an ideal fluid, that is inviscid and irrotational, even when it is largely turbulent.

Prandtl (1904) showed that the effects of viscosity are significant only in the boundary layer where the velocity gradient is not zero. Outside of this region the flow can be modelled by potentiometric methods as two-dimensional ideal flows.

There are at least two distinct layers that form sequentially, a laminar and a turbulent boundary layer. The laminar layer forms first, becomes unstable, and transitions into a composite laminar-turbulent layer. While the turbulent boundary layer tends to fill the entire profile, the laminar component shrinks to a fine “skin” usually below the typical roughness height.

The variable (laminar or turbulent) velocity V in the boundary layer approaches the uniform flow velocity U at a shallow or zero gradient:



Two key descriptions of the boundary layer are the drag force D and the layer thickness δ . These have been determined by the experimental-theoretical process as functions of a Reynolds number, Re_x :

$$Re_x = \frac{\rho U x}{\mu}$$

In Re_x the key length component is the length along the plate, not the diameter or depth of flow. This indicates that the flow develops, as indicated by the changes in depth of the layers:

$$\text{Thus shear stress} = \tau_o = C_D \rho \frac{U^2}{2} \quad \text{Skin Friction or Drag}$$

Where C_D the drag coefficient is a function of Re_x :

$$C_D = f(Re) \quad \text{Drag coefficient}$$

As is the layer depth:

$$\frac{\delta}{x} = g(Re_x) \quad \text{Boundary Layer Depth Ratio}$$

The drag per unit width of plate can be evaluated by integrating shear stress along the length to x, or by taking the average value of C_D at x and evaluating directly:

$$\text{Drag force } D = \int_0^x \tau_o dx \sim C_D \rho \frac{U^2 x}{2}$$

While this model is directly applicable only to thin plates of the type illustrated here, similar models with varying drag coefficients apply to all forms of blunt or streamlined shapes in uniform flows.

The thin plate theory is approximate in situations such as skin drag on aerofoils or hulls, and even more approximate with long rough vehicles such as train carriages and trucks. There are a few competitor models that appear in different texts, but a widely applied form of the thin plate theory (as evidenced by published texts) is a multi-phase model similar in many ways to a Moody Diagram:

Laminar Boundary Layer ($Re_x < 5 \times 10^5$)

Mean Drag Coefficient $C_D = \frac{1.328}{\sqrt{Re_x}}$

Boundary Layer Thickness $\frac{\delta}{x} = \frac{k}{\sqrt{Re_x}}$ $k \sim 4.65 \text{ to } 5.2$

Transitional Boundary Layer ($5 \times 10^5 < Re_x < 5 \times 10^6$)

Mean $C_D = \frac{0.074}{Re_x^{1/5}} - \frac{1700}{Re_x}$

$$\frac{\delta}{x} = \frac{0.38}{Re_x^{1/5}}$$

Turbulent Boundary Layer ($Re_x > 5 \times 10^6$)

a) Smooth Walls :

$$Re_x < 1 \times 10^7$$

Mean $C_D = \frac{K}{Re_x^{1/5}}$ $k \sim 0.059 \text{ to } 0.074$

$$\frac{\delta}{x} = \frac{0.38}{Re_x^{1/5}}$$

$$Re_x > 1 \times 10^7$$

Mean $C_D = \left(\frac{1}{4.13 \ln(Re_x C_D)} \right)^2$

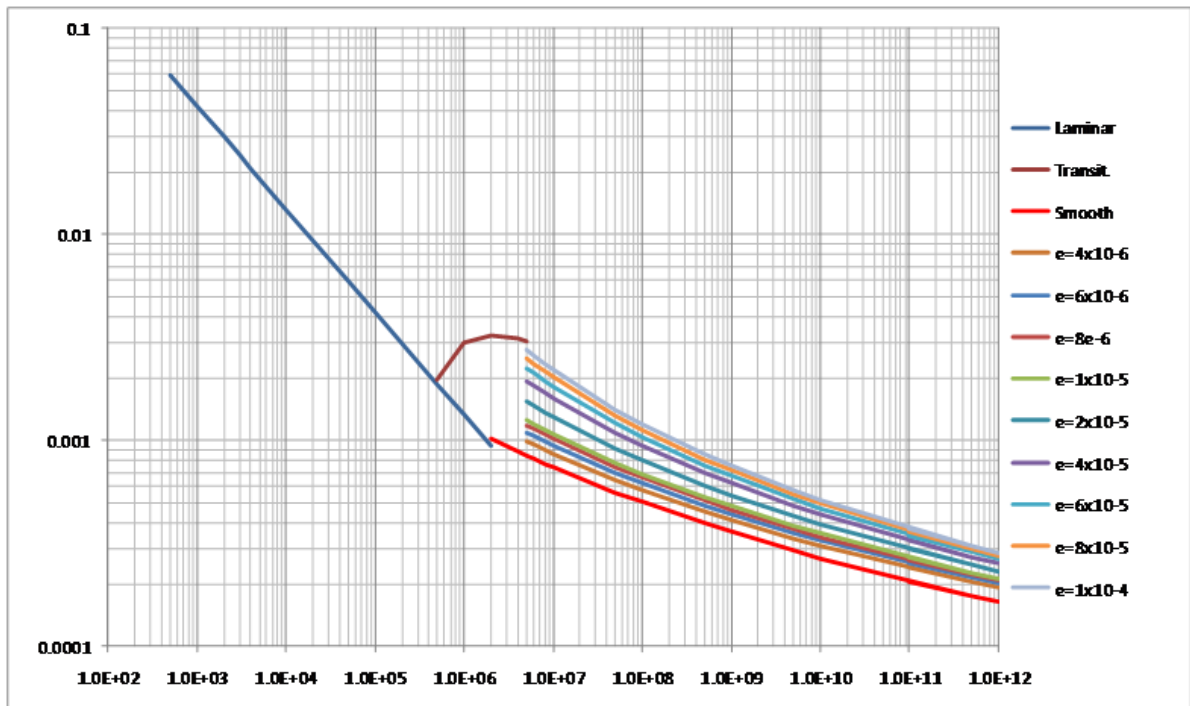
$$\frac{\delta}{x} = \frac{0.22}{Re_x^{1/6}}$$

b) Rough Walls:

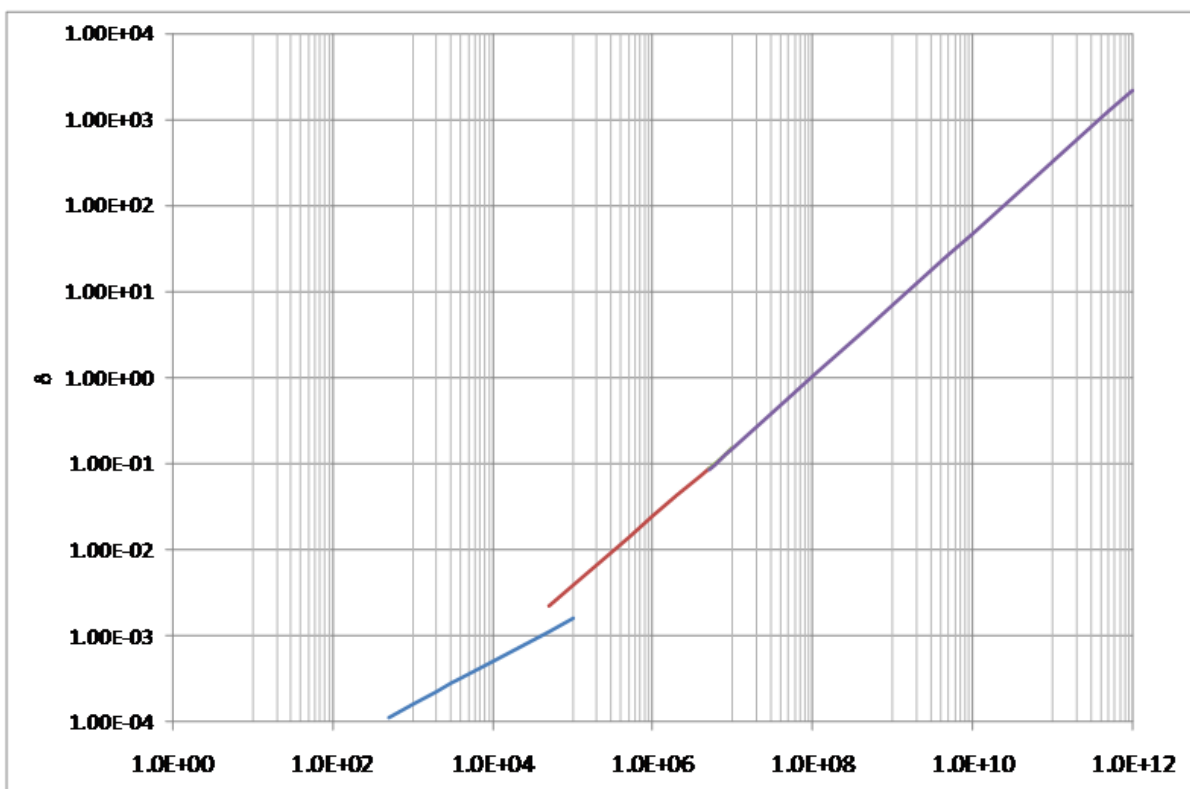
$$C_D = \left(\frac{1}{3.96 \ln(\delta/\epsilon) + 7.55} \right)^2$$

(ϵ = Roughness Height)

$$\frac{\delta}{x} = \frac{0.22}{Re_x^{1/6}}$$



Plot of thin plate drag coefficient C_D against Reynolds number. For convenience the fully-developed smooth wall turbulence range is extended to the laminar and the minor smooth wall range is ignored.



Plot of thin plate boundary layer depth (m) against Reynolds Number. Note the step at L-T transition.

These equations and charts are approximate for most real applications and should be used only in a generic design sense. More accurate estimates of drag may require experimental measurements over the specific design and operating range. The key equations in the design process are highlighted on the previous page.

Parameterised Logarithmic Velocity Distribution

It is now possible to estimate the shear velocity and hence parameterise the logarithmic velocity distribution:

We have:

$$\tau_o = C_D \rho \frac{U^2}{2} \quad \text{and} \quad u^* = \sqrt{\frac{\tau_t}{\rho}} = \sqrt{\frac{\tau_o}{\rho}}$$

Thus $u^* = U \sqrt{\frac{C_D}{2}}$

And $V = \frac{U}{k} \sqrt{\frac{C_D}{2}} \ln\left(\frac{y}{\delta}\right) + c$ $k \sim 0.4$; C_D , δ as defined **Turbulent Velocity Distribution**

This leaves the integration constant c to be evaluated in particular cases. For $y \ll \delta$ unfortunately the model produces large negative values, going to $-\infty$ at $y = 0$.

Two commonly cited calibration formulae are (for $R_x > 5 \times 10^5$):

$$V = 5.75 u^* \ln\left(\frac{u^* y}{\nu}\right) + 5.5 \quad \text{Smooth-wall turbulence, } \nu = \text{kinematic viscosity}$$

$$V = 5.75 u^* \ln\left(\frac{y}{\varepsilon}\right) + 8.5 \quad \text{Rough-wall turbulence, } \varepsilon = \text{roughness height}$$

Prandtl also identified an approximate $1/7^{\text{th}}$ power law in the smooth-wall turbulent velocity distribution. This leads to similar values of the drag coefficient and is a simpler alternative with no negative flows or risk of infinities at $y = 0$:

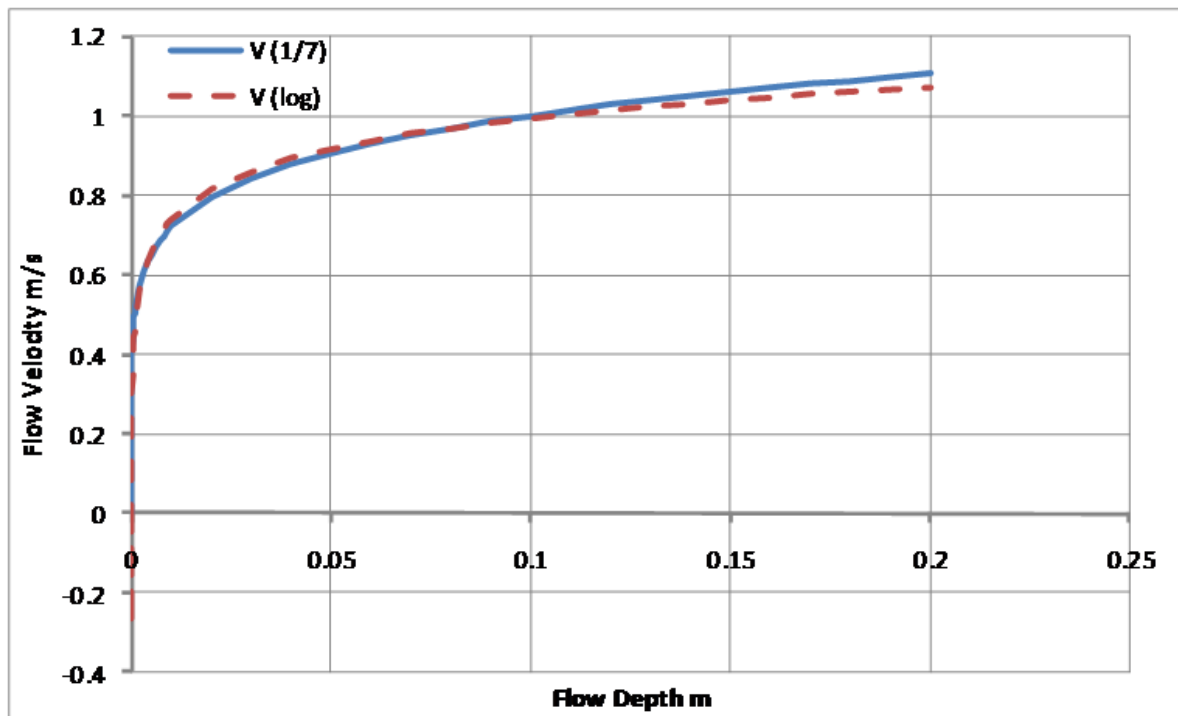
$$V = U \left(\frac{y}{\delta}\right)^{1/7} \quad \text{Prandtl's } 1/7^{\text{th}} \text{ Power Velocity Distribution for Turbulent Smooth, Flat Plates}$$

$$V = U \left(\frac{y}{R}\right)^{1/7} \quad \text{Prandtl's } 1/7^{\text{th}} \text{ Power Velocity Distribution for Turbulent Pipes of Radius } R$$

So do we use the logarithmic or the $1/7^{\text{th}}$ distribution? What do we do with alternative models?

- Try them both, varying parameters to find best comparative fit
- Compare and calibrate them to your data;
- Decide/evaluate which gives the best fit to the data, and/or
- Use the insights to develop them further.

See the comparison below for example:



Comparison of $1/7^{\text{th}}$ -power and logarithmic velocity distributions for a flat plate turbulent boundary layer with free stream velocity $U = 1 \text{ m/s}$ @ 100 mm flow depth. The main difference between these models is the negative velocities at very shallow depths in the logarithmic model. Note that neither model predicts a fixed upper value of U , however the logarithmic model increases more slowly.

Drag Coefficient for Laminar and Turbulent Pipe Flow

A drag coefficient for general pipe flow can be derived from the pipe friction factor f :

As before we have:

$$\tau_o = C_D \rho \frac{U^2}{2}$$

Also $S_f = \frac{\tau_o}{\rho g R}$ Friction Slope Equation

Where $R = \text{Hydraulic Radius} = A/P = \frac{\pi D^2}{4\pi D} = \frac{D}{4}$

And $S_f = f \frac{V^2}{2gD}$ Darcy-Weisbach Equation

Thus $\tau_o = \rho g R S_f = \rho f \frac{V^2}{8} = C_D \rho \frac{V^2}{2}$

So $C_D = \frac{f}{4}$ General Pipe Flow Drag Coefficient

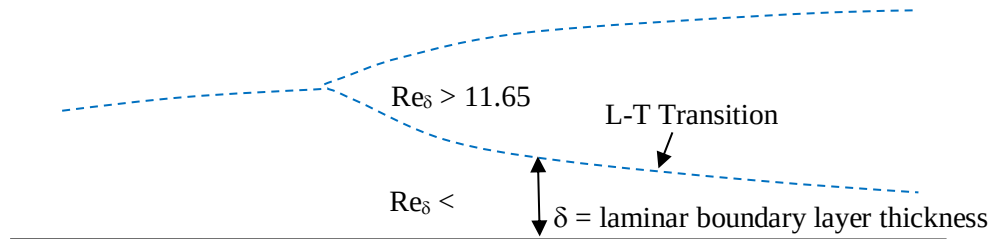
Where f is defined as before (use Moody diagram or three-phase model). US texts use the symbol λ in place of f for friction factor, often with a conversion factor of $1/4$. Watch out for that!

Composite Laminar – Turbulent Transitions

Another type of Reynolds number governs the transition from laminar to turbulent in the composite laminar-turbulent boundary layer:

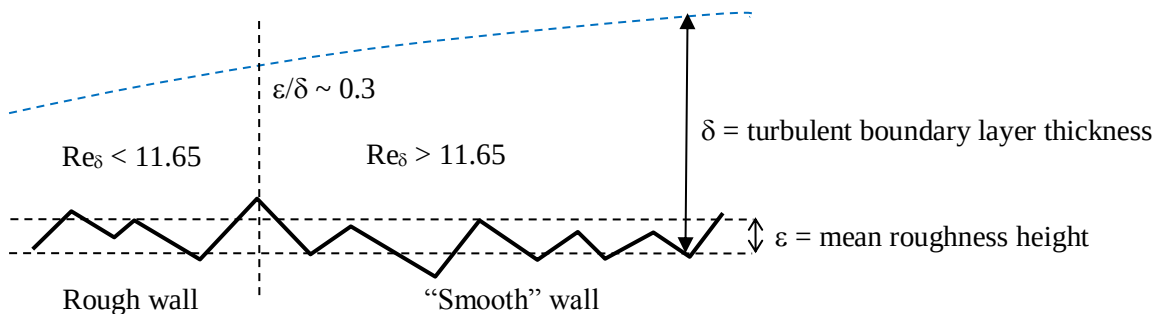
$$Re_{\delta} = \frac{\rho u^* \delta}{\mu} = \sqrt{\frac{C_D}{2}} \frac{\rho U \delta}{\mu} \sim 11.65 \text{ at composite transition}$$

The composite layer transitions from laminar to turbulent at δ such that $Re_{\delta} \sim 11.65$



Rough – Smooth Wall Turbulent Transitions

The same Reynolds number occurs at the transition from rough to smooth wall turbulence: Wall roughness is relative to the thickness of the boundary layer, ie a boundary layer that has developed to be significantly thicker than the roughness height begins to behave as per smooth wall turbulence. This occurs roughly at $\epsilon/\delta \sim 0.3$, which again corresponds to $Re_{\delta} \sim 11.65$:



Reading

Streeter and Wylie 1981 (Reading 10) Section 5.6 and 5.8