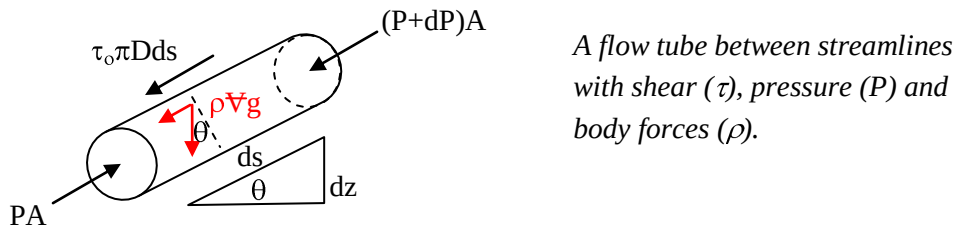


Lecture 20: Shear Stress and Headloss

Section 4 introduced a number of concepts relating to shear stress as components of drag and headloss. In this section we examine the linked experimental-theoretical process by which this important physical idea was developed and formulated.

Restricting the analysis initially to uniform steady flow, we can find a simple relationship between shear stress and headloss:



Velocity and area are constant, forces are in balance, no acceleration:

$$PA - \rho g A ds \sin \theta - (P + dP)A - \tau_0 \pi D ds = 0$$

Forward pressure
Back component of weight
Back pressure
Shear drag on cylindrical surface

Dividing by $\rho g A ds$ and note $\sin \theta = dz/ds$,

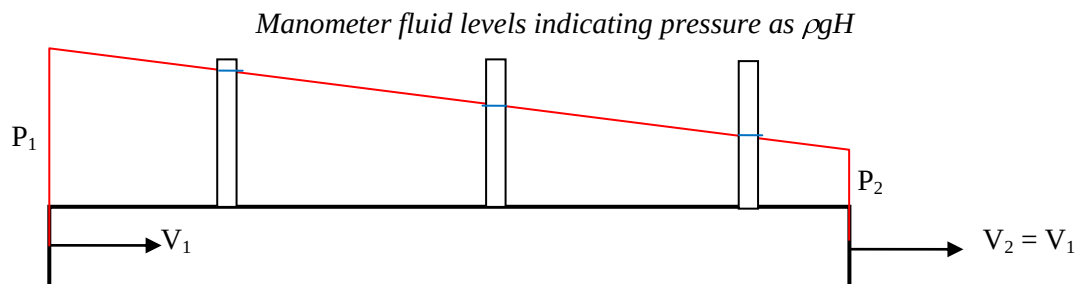
$$-\frac{d}{ds} \left(\frac{P}{\rho g} + z \right) - \frac{\tau_0}{\rho g R} = 0$$

where R is the *Hydraulic Radius* $R = \text{Area}/\text{Wetted Perimeter} = \pi D^2 / \pi D = D/4$

$$A = \pi D^2 / 4 \quad \text{Pipe cross section} \quad P = \pi D \quad \text{Hydraulic Radius } R = A/P = D/4 = r/2$$

But $-\frac{d}{ds} \left(\frac{P}{\rho g} + z \right)$ is the slope of a steady hydraulic grade line (ie the P and z terms of Bernoulli).

In the previous section we saw that a sloping HGL is associated with head loss. This slope is referred to as the *friction slope* or *hydraulic gradient* S_f .



Thus

$$\frac{dh}{ds} = \frac{h_f}{L} = \text{Friction slope} = S_f = \frac{\tau_0}{\rho g R} \quad \text{Friction slope equation}$$

Where τ_0 is the shear stress due to viscous friction at the wall boundary.

The friction slope equation also applies to non-uniform flow. The headloss in units of height (m) over a given length is thus

$$h_f = -\Delta \left(\frac{V^2}{2g} + \frac{P}{\rho g} + z \right) = \frac{\tau_0 L}{\rho g R}$$

In uniform flow situations the velocity is constant and so the sum of pressure and elevation terms (the piezometric head) declines. It remains to determine the shear stress τ_0 :

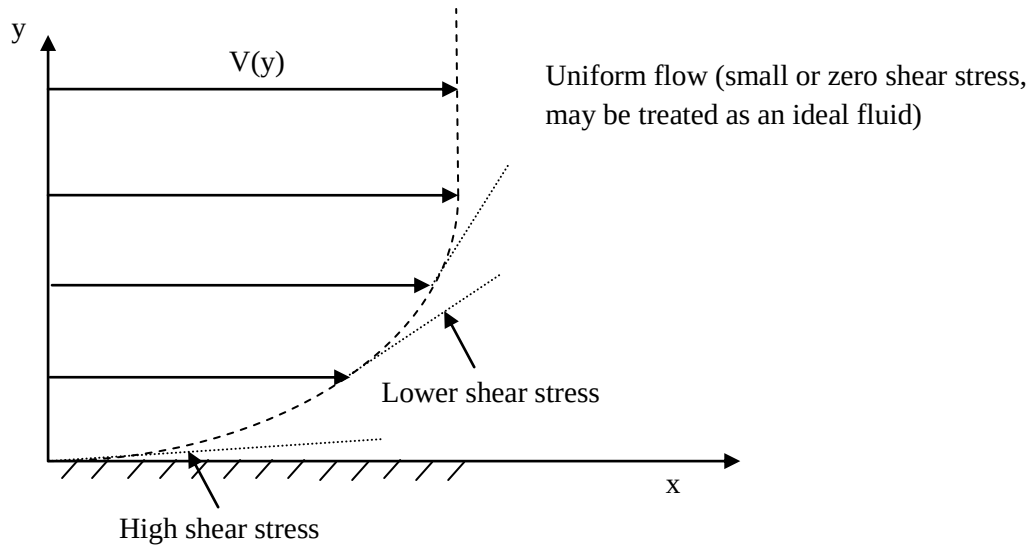
Shear Stress: Viscous and Turbulent Friction

The layer shear stress τ_0 is associated with friction, which may be due to viscous adhesion or the turbulent transfer of momentum:

Laminar Viscous Friction

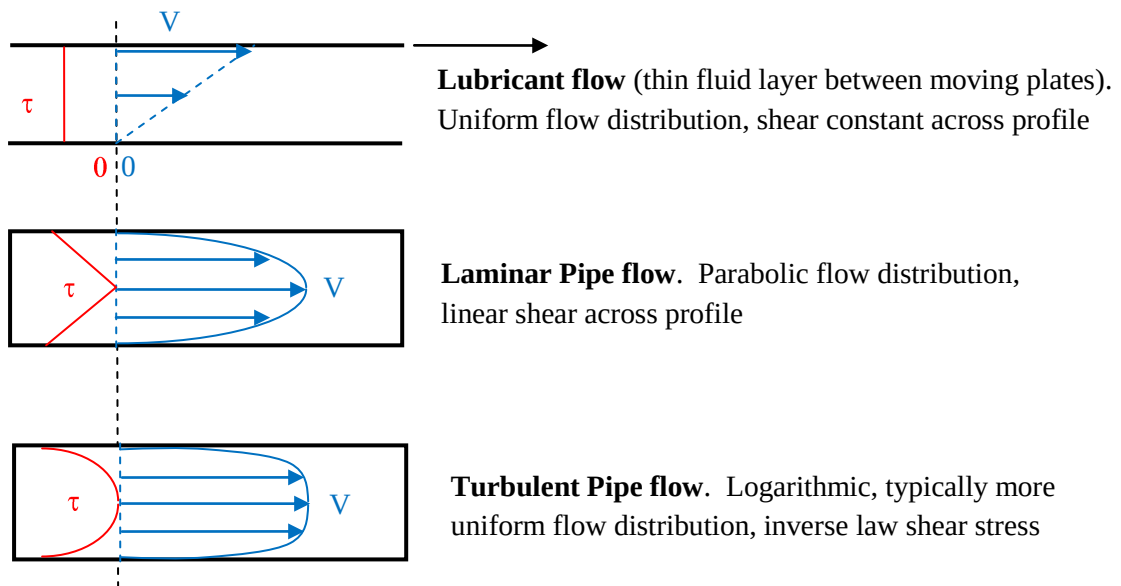
Newton: $\tau = \mu \frac{dV}{dy}$

Viscosity is a constant proportionality between the flow profile and the resulting dynamic shear stress:

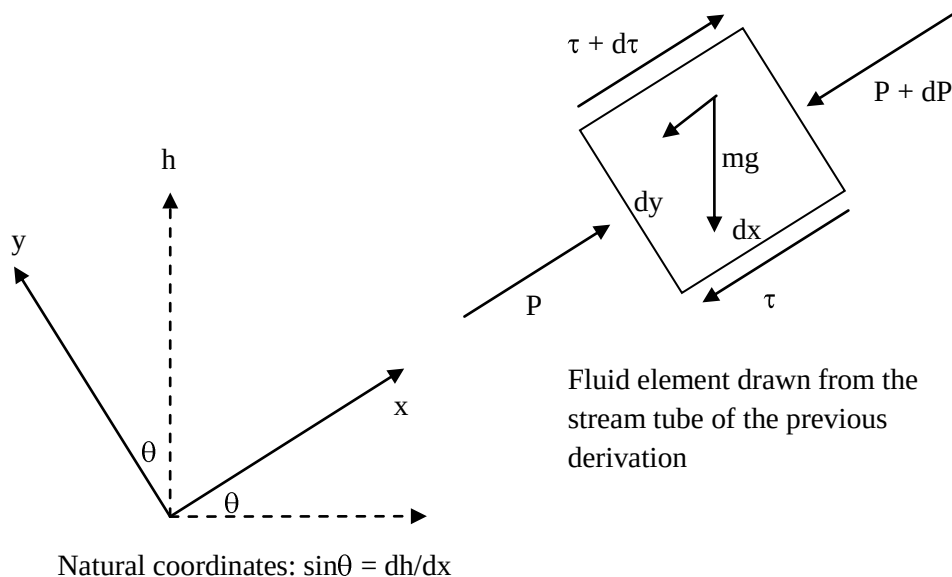


The wall shear stress is typically the maximum, ie $\tau_0 = \mu \frac{dV}{dy} \Big|_{y=0}$.

The shear stress and hence friction slope thus depend on the velocity slope at the wall, bed or boundary, which is observed to take different profiles depending on the type of boundary or flow regime:



To determine the type of velocity profile we need to examine incremental changes in pressure *and* shear along streamlines:



Steady uniform flow: $\Sigma F_x = 0$: per unit width,

$$Pdy - (P + dP)dy - \tau dx + \tau + d\tau dx - \rho g dx dy \sin\theta = 0$$

ie $-dPdy + d\tau dx - \rho g dx dy dh/dx = 0$

Divide by $dx dy$ and rearrange:

$$\frac{d\tau}{dy} = \frac{d}{dx}(P + \rho gh) \quad \text{Fundamental Solution of Navier-Stokes Equation}$$

This solution is an equivalent form of the friction slope equation, with the additional detail of a varying shear stress profile in the derivation. It gives further information on the velocity profile, since:

$$\frac{d\tau}{dy} = \frac{d}{dy}\left(\mu \frac{dV}{dy}\right) = \mu \frac{d^2V}{dy^2} \quad \text{Second derivative of velocity profile.}$$

We can thus determine velocity profiles through observation of the slope of the pressure gradeline. The results depend on boundary conditions specific to particular cases.

Two useful cases are 1) lubricant flows and 2) pipe flows:

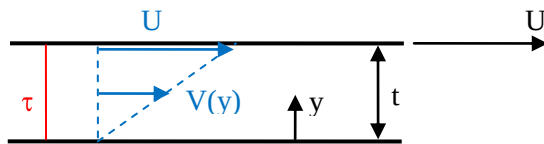
Lubricant Flow (Taylor-Couette Flow)

Lubricants flow at nearly constant pressure between moving plates. In an idealised setting, a plate moving in one direction rides on a cushion of constant pressure supported by the other plate.

Thus
$$\frac{d(P + \rho gh)}{dx} = 0 = \frac{d\tau}{dy}$$

And since $\frac{d\tau}{dy} = 0$, $\tau_y = \text{constant } k$.

But if $\tau = \mu \frac{dV}{dy} = k$, then $V(y) = ky$ is a linear velocity profile as observed (eg Newton).



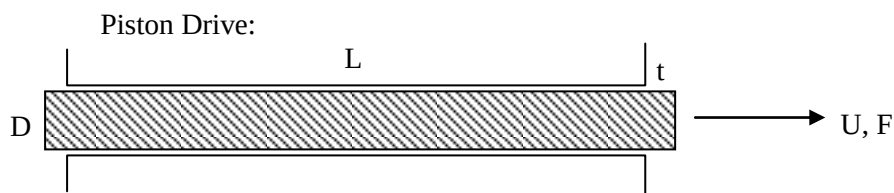
$$V(y) = U \frac{y}{t} \quad \frac{dV}{dy} = \frac{U}{t} \quad \tau = \mu \frac{U}{t} \quad \bar{V} = \frac{U}{2} \quad q = \bar{V}t = \frac{Ut}{2} \quad \text{Lubricant equations}$$

q = lubricant flowrate per unit width

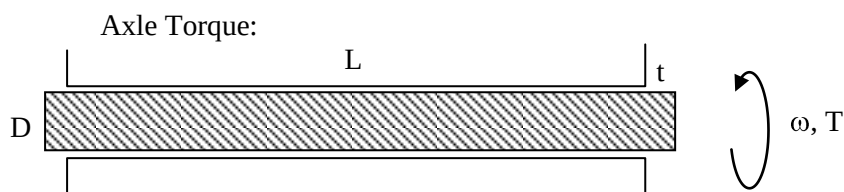
The force required to produce the plate speed U is

$$F = \tau A = \frac{\mu U A}{t}$$

where A is the plate contact area. This can be generalised to specific lubrication arrangements:

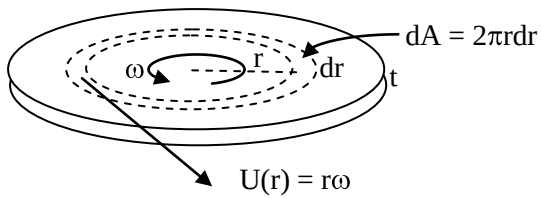


$$\tau = \mu \frac{U}{t} \quad F = \tau \pi D L \quad \text{Power} = F U \quad \text{Additional power to force lubricant.}$$



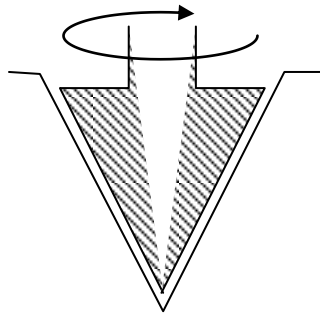
$$\tau = \mu \frac{r \omega}{t} \quad T = \tau \pi D L r \quad \text{Power} = T \omega$$

Disk (eg clutch plate) lubrication:



$$\tau(r) = \mu \frac{r\omega}{t} \quad dF = \tau dA \quad dT = r dF \quad T = \int r \mu \frac{r\omega}{t} 2\pi r dr = \mu \omega \frac{2\pi}{t} \int_0^R r^3 dr$$

Conical fittings



? Try it for the exercise

Reading

Streeter and Wylie 1981 (Reading 10) Section 5.1