

Supplementary Lecture 2: More Topics in Dimensional Analysis

The Buckingham Pi Technique

One of the most basic ideas in mathematical physics is that all the additive terms of an equation must have the same units. You can't add apples and oranges unless you include them both in a higher category in which they are equivalent, such as dietary energy content, or mass, or cost value.

Because of this, in many cases you can work out the form of an equation by ensuring that it has the right units: the quantity you are trying solve for, be it a force or a velocity, must have the units of force or velocity by combinations of variables in every additive term of the formula.

Deriving by units is not a physical derivation of a formula, which normally involves physical *and* mathematical reasoning, with well-chosen simplifications that capture the key details or ignore extreme ranges. As a last step, simple dimensional analysis (re-writing a formula into units) can be applied as a way of checking that your derivation makes sense and can possibly be correct.

The Buckingham Pi technique¹ is a more formal method that sets out from the first stage of a physical derivation, that is the stage of identifying the physical variables that an effect depends upon (such as force depending on mass and acceleration) and then examines those variables in a rigorous analysis that yields all possible functions of those variables that can produce the effect. The result is a set of dimensionless functional groups that can then be tested experimentally to show, as simply as is possible, the form of an effect in the most general range of cases.

Massey, 2012: "Where a number of independent variables affect a dependent variable, all the variables relate to each other, not in a haphazard manner but in a highly structured way... Because the individual variables in a dimensionless group must behave exactly as the whole group, it is only necessary to vary one of the independent variables systematically."

This is clearly a very powerful technique, and not limited to fluid mechanics.

¹ Often referred to as the Buckingham Pi Theorem, except that Buckingham did not prove the theorem but developed it into a research method (circa 1914). See Bertrand, J. (1878). "Sur l'homogénéité dans les formules de physique". *Comptes rendus*. 86 (15): 916–920.

The General Form of Physical Equations

A physical equation generally states that some *dependant* physical variable is a mathematical function of other *independent* variables, which could be experimentally varied, one at a time, to test how the dependent variable varies in response to each of its arguments.

For instance the drag force on a blunt object in a stream of fluid depends on the size and shape of the object, the density and viscosity of the fluid, and the velocity of the flow. Similarly the frictionless headloss in a pipe depends on the length, diameter and internal roughness of the pipe, the density and viscosity of the fluid, and the velocity of the flow.

The drag and friction force laws combine all those variables by mathematical operations that convert their numerical and dimensional values into the numerical and dimensional values of forces.

This seemingly obvious statement of quantitative physics is often complicated by the existence of multiple effects controlling dependancies between the independent variables, so that not every range of independent variables is physically possible, but many different combinations will produce the same effect.

Dimensionless numbers are the most important of these types of combinations since, as we have seen, a range of variables can be independently varied over many scales to give the same dimensionless value, which then produces the same effect for any range of variations that have that numerical value.

The Buckingham Pi technique (set out below) shows that the most general form of a physical equation is a dimensionless form of the dependent variable, written as a function of other dimensionless forms of the independent variables.

The real power of the B-Pi technique is that you don't have to experimentally vary the values of all the variables, just the experimental value of each dimensionless combination. So if you have five or six independent variables, you can combine them into one or two dimensionless groups, and then you only need to plot your data along one or two axes. Often it's possible to combine all the dimensionless groups into a single graph, which then applies to every possible case, at any scale, in any fluid.

The drag force on a blunt object in dimensionless form is

$$\frac{F_D}{\rho V^2 D^2} = f\left(\frac{\rho V D}{\mu}\right)$$

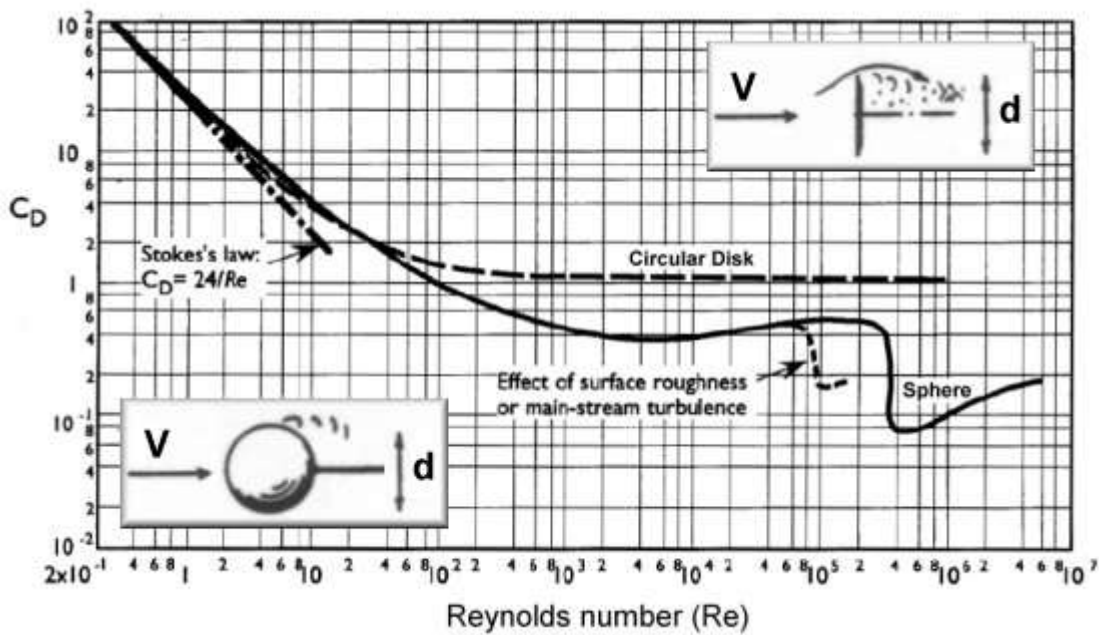
Experiments on drag force find that dimensionless force (check that the LHS is dimensionless) is predicted by a particular function of Reynolds number that appears in all experiments regardless of the size of the object or the properties and parameters of the flow. This function is graphed from experimental data on the next page.

The function f is a form of *drag coefficient* (see graph), with some additional constant values so that $f = C_D \pi / 8$. The theoretical form of C_D is unknown but it very simply predicts the drag force. We can rewrite this important drag force law as:

$$F_D = C_D \frac{\rho V^2}{2} A$$

Where $A = \pi D^2 / 4$ is the profile area of a sphere, the basic object examined in drag studies. The dependance of force on Reynolds number, and hence the four independent variables V , D , ρ and μ (separate from the important dimensional group $\rho V^2 / 2$, the dynamic pressure), is captured by a single graph of the drag coefficient:

Drag Coefficient

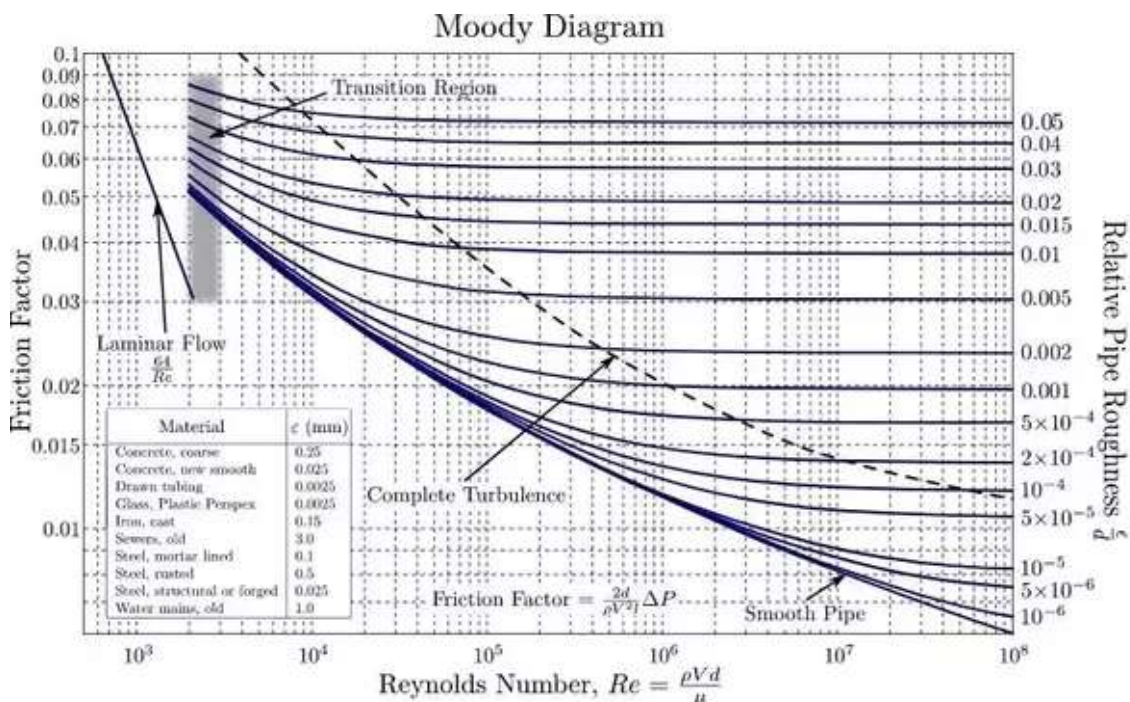


There is no complete theory to explain the functional form of the drag coefficient. Fortunately we don't need theory to show experimentally that the drag coefficient evaluates as shown in the figure

Similarly the headloss due to friction (h_f) in a long (L) pipe (D) with roughness height (ε) at flow velocity (V) can be non-dimensionalised as

$$\frac{h_f}{\left(\frac{LV^2}{gD}\right)} = f\left(\frac{VD\rho}{\mu}, \frac{\varepsilon}{D}\right) \quad \text{or} \quad h_f = f \frac{LV^2}{2gD}$$

where f is a *friction factor* of both the Reynolds number and the relative roughness of the pipe:



In both cases the value of $\frac{1}{2}$ identifies the force and head laws as kinetic energy. In both cases a single graph is all you need to know about the physics of fluid friction.

Buckingham Pi is useful most of all because it reduces the algebraic process of developing theory with experiment to a few repeatable steps, that have a few learnable subtleties.

I. Identify Relevant Physical Variables

The first step is to identify the variables that a physical effect depends on. This requires background in the physical problem under investigation, usually with prior experimentation that may have included or eliminated variables that the effect does not depend on.

The selection of relevant physical variables may proceed by trial and error, since in fact the B-Pi method doesn't work and leads to contradictions if too many or too few variables are included.

Variables in fluid mechanics are of three types:

1. **Geometry:** Length, Area, Volume, Shape, Roughness (fine texture geometry)...
2. **Fluid Properties:** Density, Viscosity, Vapour Pressure, Surface Tension...
3. **Conditions:** Velocity, Acceleration, Pressure, Temperature, Humidity...

There are few problems in fluid mechanics that don't use variables of all three types. Ensure you have given thought to all three categories.

II. Identify the Dimensions of variables

Physics has only five "dimensions" that cannot be reduced to other dimensions. These are

1. Mass **M** in units of kilograms [kg]
2. Distance **L** in units of metres [m]
3. Time **T** in units of seconds [s]
4. Temperature **Φ** in units of kelvin [K]
5. Electric Current **I** in units of amperes [A]

In fluid mechanics only the first four are needed, and only the first three in cases of incompressible flow. Fluids are thus often referred to as three-dimensional MLT problems. In some texts, Mass M is replaced with Force F, which is common and simplifies the units of mass and acceleration where they occur together. We'll stick to MLT, for the practice of reducing to the more fundamental mass units.

III. Recall the Theorem (Bertrand, J, 1878)

If a physical process involves N dimensional variables (physical quantities, properties or conditions) that operate in J dimensions, then there are at most $K = N - J$ dimensionless variables that must be experimentally varied to completely specify the process.

So for example the drag force has five variables including itself (force, velocity, diameter, density, viscosity), with three dimensions MLT, leading to no more than two dimensionless numbers to explore in the lab. These are present on both sides of the dimensionless form of the equation:

$$\frac{F_D}{\rho V^2 D^2} = f\left(\frac{\rho V D}{\mu}\right)$$

Often the value of K turns out to be less than N - J, depending on how you simplify the problem. Drag force for instance is simplified by considering ideally smooth blunt bodies (ignores skin friction) of just one simple shape, a sphere (ignores all shape-dependent effects to a good approximation). As the C_D graph shows, different shapes and surface roughnesses have different drag coefficients, but you

can develop graphs for any particular shape and roughness (such as a streamlined kevlar-shell hull design) that is just one function of Re .

Step V simply tells you how many times (K) you will need to repeat the main technique, which is to combine your variables into groups and use linear algebra to determine the power laws that must occur in the process.

IV. Select J Dimensional “Repeating Variables” from all Independent variable types.

Let’s summarise the drag force analysis so far to see what this means:

We have

	Dependent	Independent			
		Geometry	Condition	Fluid Properties	
Variable:	Force	Diameter	Velocity	Density	Viscosity
Symbol	$F [kgms^{-2}]$	D	V	ρ	$\mu [Pa.s]$
Dimensions	MLT^{-2}	L	LT^{-1}	ML^{-3}	$ML^{-1}T^{-1}$

$N = 5$ variables with 4 independent; $J = 3$ dimensions and 3 repeating variables; $K \leq 5 - 3 = 2$ dimensionless groups

We need to select 3 independent variables, representing geometry, properties and conditions, that have among them all the dimensions of the effect under study, but that cannot form a dimensionless group by themselves. This last point is perhaps the trickiest: It must not be possible to multiply your chosen variables in any combination, raised to any power, that reduces to a dimensionless number.

This is in effect the same as looking for the dimensionless numbers, the whole point of the exercise, but easier because there are many more dimensional combinations than dimensionless ones.

At the simplest this would mean choosing variables that are all of entirely separate dimensions M , L and T (such as mass, diameter and frequency), since then you can’t raise any one to a power that cancels any other.

Exercise: Show that there is no dimensionless combination of mass m , diameter d and frequency $1/t$.
Try raising these variables to various powers and multiply or divide in different combinations.

In practice, you can select variables that are dimensionally “sparse”, that is each contains as few dimensions as possible, allowing the other chosen variables to carry the other dimensions, so that some random combination cannot cancel the dimensions.

In this case, D , V and ρ represent geometry, fluid properties and flow conditions, and are sparse, since only V has the time dimension T and only density has the mass dimension M . An alternative V , ρ and μ is wrong because the three variable types are not represented and they are not sparse: the dimensions are present in pairs, which will probably cancel in some combinations.

Having selected J repeating variables, we expect there to be K power-law combinations with the remaining variables that will reduce to dimensionless numbers, which we can now find by routine:

V. Write repeating variables as power laws with the remaining $K = N - J$ variables

This just means writing the J repeating variables, raised to unknown powers, in combination with the other variables, to form K combinations. Buckingham contributed the notation Π to denote these numbers, counted from 1 to K :

$$\Pi_1 = D^a V^b \rho^c \mu = [M^0 L^0 T^0]$$

$$\Pi_2 = D^p V^q \rho^r F = [M^0 L^0 T^0]$$

The powers or indices a, b and c , then p, q and r , are to be solved by adding them to zero, since we are multiplying terms, with the effect of *adding* indices.

VI. Rewrite the equations from step V as dimensions, and then as Index Sums:

For every variable in step V we can write its dimensional form raised to a power:

$$\Pi_1 = [L]^a [LT^{-1}]^b [ML^{-3}]^c [ML^{-1}T^{-1}] = [M^0 L^0 T^0]$$

For every dimension there is a series of powers adding to zero. Looking at the indices on each dimension in turn:

$$L: [L]^a [L]^b [L^{-3}]^c [L^{-1}] = [L^0]$$

$$\text{or } L: a + b - 3c - 1 = 0$$

$$\text{also } T: -b - 1 = 0$$

$$\text{and } M: c + 1 = 0$$

You can then use linear algebra to solve for the powers. Often (if your repeating variables are sparse enough) you can see the solution easily:

$$c = -1$$

$$b = -1$$

$$a + (-1) - 3(-1) - 1 = 0 \rightarrow a = -1$$

Which gives us the first dimensionless group:

$$\Pi_1 = D^{-1} V^{-1} \rho^{-1} \mu$$

Often you can invert or simplify:

$$\Pi_1 = DV\rho\mu^{-1} = \frac{DV\rho}{\mu} = Re$$

So we find the Reynolds number naturally enough

Exercise: Solve for Π_2 by the same methods in Step VI. You can reuse the superscripts a, b, c so long as you don't mix them up between Π_1 and Π_2

As expected, you should have found

$$\Pi_2 = \frac{F_D}{\rho V^2 D^2}$$

VII. Write Dependent and Independent Dimensional Groups as a Function

The last theoretical step of the process is to write the derived equations both non-dimensionally and dimensionally for the dependant variable you set out to model:

$$\frac{F_D}{\rho V^2 D^2} = f\left(\frac{\rho V D}{\mu}\right)$$

$$F_D = \rho V^2 D^2 f\left(\frac{\rho V D}{\mu}\right)$$

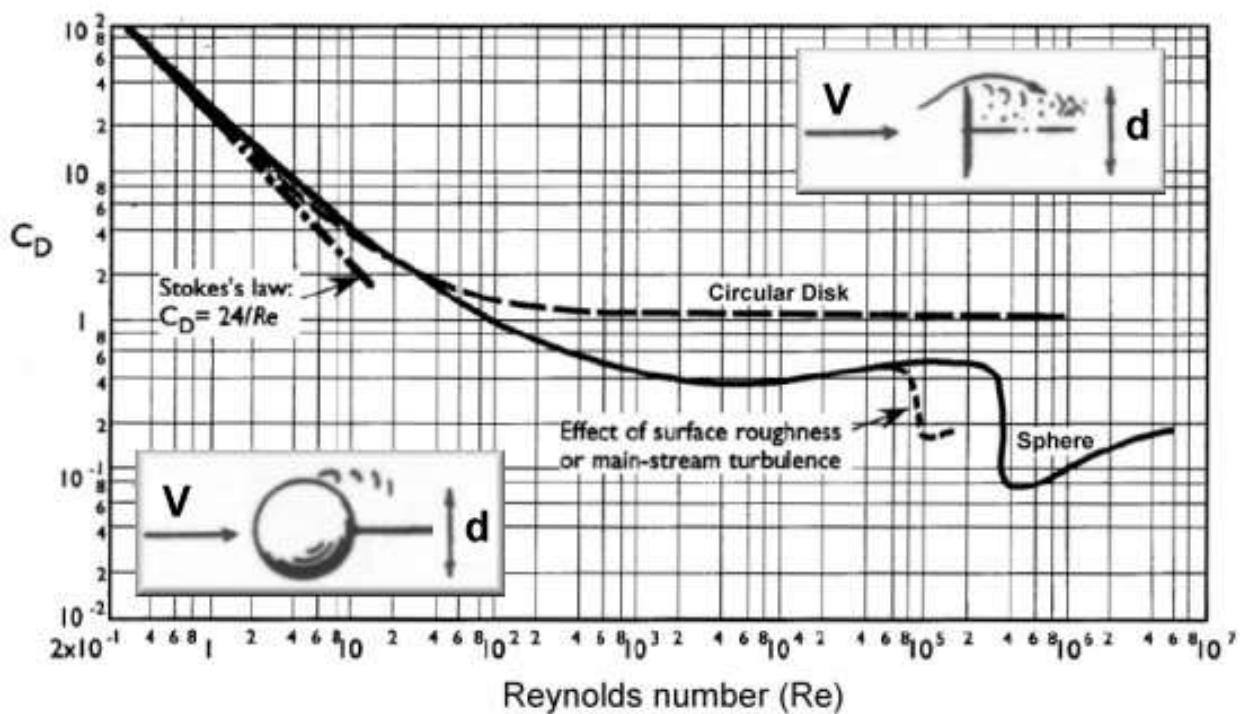
VIII. Experiment with the Dimensionless Groups, not the variables

This last step is experimental, to plot data on axes defined by these non-dimensional parameters and obtain a graph of the function. We then don't need to know the theoretical form of the function, just that it has a certain experimental shape and values over ranges of the dimensionless numbers.

By convention, the drag formula has evolved into a statement about the profile area $A_p = \pi D^2/4$, the dynamic pressure $P_d = \rho V^2/2$, and a dimensionless C_D that is published consistent with

$$F_D = C_D \frac{\rho V^2}{2} A_p$$

where A_p is the projected area of the blunt object and the data used for C_D is processed to include the dimensionless multiplier $\pi/8$. The experimental process leading to a graph of C_D requires only a systematic variation of the value of Reynolds number by any means, and not the more complex task of systematically varying *all* the independent variables.



This is a very useful and commonly used formula, yet we don't have a complete theoretical derivation to explain it. We don't need one! But it will be interesting when someone finds it...

Let's try that again with frictional pressure loss in a pipe:

Pipe friction head loss has six variables including itself (headloss, velocity, diameter, density, viscosity, and relative roughness) in three dimensions, hence there are three dimensionless numbers including the left hand side dependent variable group (check that the LHS is dimensionless):

$$\frac{h_f}{\left(\frac{LV^2}{gD}\right)} = f\left(\frac{VD\rho}{\mu}, \frac{\varepsilon}{D}\right)$$

Pressure loss due to friction P_f is a function of:

Geometry: Pipe length L , diameter D and roughness height ε .

Properties: Fluid viscosity μ and density ρ .

Conditions: Velocity V .

	Dep:	Indep:					
Var:	Pressure	Length	Diameter	Roughness	Viscosity	Density	Velocity
Sym:	P_f	L	D	ε	μ	ρ	V
Dim:	$ML^{-1}T^{-2}$	L	L	L	$ML^{-1}T^{-1}$	ML^{-3}	LT^{-1}

$N = 7$ variables with 6 independent; $J = 3$ dimensions and 3 repeating variables highlighted;

$K \leq 7 - 3 = 4$ dimensionless groups

We'll use the proven repeating variables $D\rho V$ again.

$$\Pi_1 = \rho^a V^b D^c L$$

$$\Pi_2 = \rho^a V^b D^c \varepsilon$$

$$\Pi_3 = \rho^a V^b D^c \mu$$

$$\Pi_4 = \rho^a V^b D^c P_f$$

$$\Pi_1 = M^a L^{-3a} L^b T^{-b} L^c L = M^0 L^0 T^0$$

$$M: \quad a = 0$$

$$L: \quad -3a + b + c + 1 = 0$$

$$T: \quad -b = 0$$

$$L: \quad c = -1$$

$$\Pi_1 = \rho^0 V^0 D^{-1} L = \frac{L}{D}$$

This makes as a dimensionless number. Π_2 has the same units, since $[\varepsilon] = [L] = L$

$$\text{So} \quad \Pi_2 = \rho^0 V^0 D^{-1} \varepsilon = \frac{\varepsilon}{D} \quad \text{Relative roughness ... OK}$$

Π_3 can only resolve as Reynolds number, which is the only dimensionless combination of $D\rho V$ and μ .

$$\text{So} \quad \Pi_3 = \frac{DV\rho}{\mu} = Re$$

The last Π_4 will be the dependent dimensionless group:

$$\begin{aligned}\Pi_4 &= \rho^a V^b D^c P_f \\ &= M^a L^{-3a} L^b T^{-b} L^c M L^{-1} T^{-2} = M^0 L^0 T^0\end{aligned}$$

$$M: \quad a + 1 = 0 \quad a = -1$$

$$L: \quad -3a + b + c - 1 = 0$$

$$T: \quad -b - 2 = 0 \quad b = -2$$

$$L: \quad 3 - 2 + c - 1 = 0 \quad c = 0$$

So $\Pi_4 = \rho^{-1} V^{-2} D^0 P_f = \frac{P_f}{\rho V^2}$ This is the Euler number. Must be on the right track.

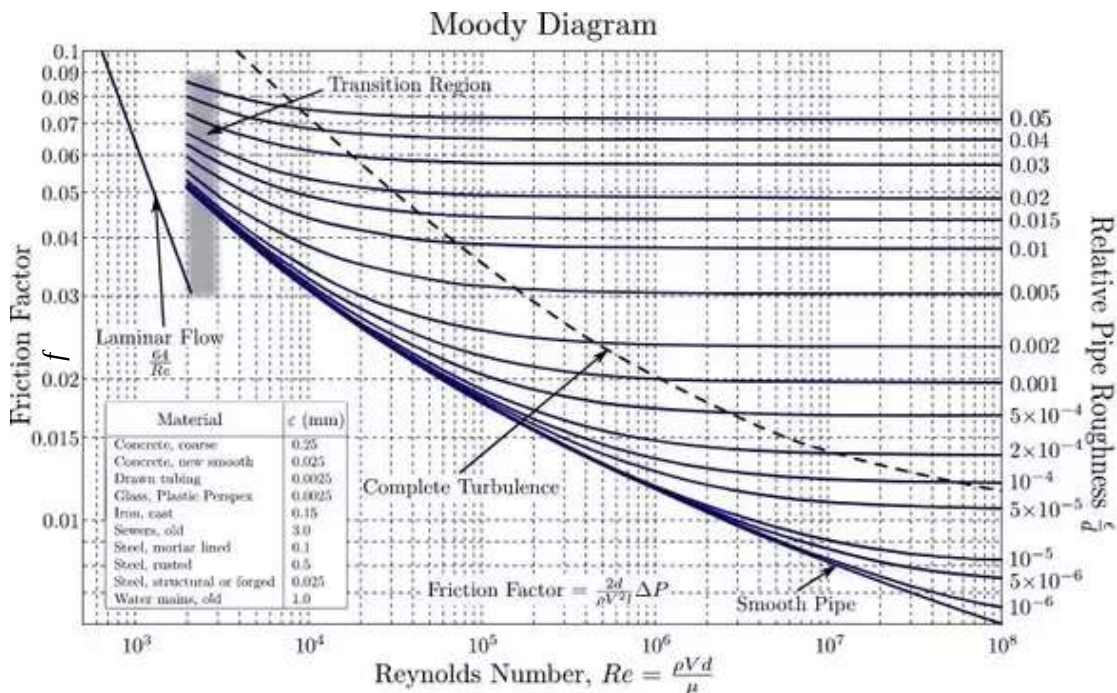
Collecting the dimensionless form of the formula,

$$\frac{P_f}{\rho V^2} = f \left(\frac{L}{D}, \frac{\varepsilon}{D}, \frac{DV\rho}{\mu} \right)$$

This equation was explored experimentally over several decades and is now better known in a form referred to as the Darcy-Weisbach equation², with a graphical solution due to Nikuradse and Moody. Generally it is presented as a headloss equation:

$$h_f = \frac{P_f}{\rho g} = f \frac{L}{D} \frac{V^2}{2g} \quad \text{Darcy-Weisbach Equation of pipe friction loss}$$

Where $f = f \left(\frac{\varepsilon}{D}, \frac{DV\rho}{\mu} \right)$ is completely specified by the graph:



For given values of Re and ε/D the friction factor f and headloss h_f are completely specified.

² Darcy, H. 1857. "Experimental Research Relating to the Movement of Water in Pipes" Mallet-Bachelier, Paris. 268 pages and Weisbach, J., 1845. Lehrbuch der Ingenieur- und Maschinen-Mechanik, Braunschweig

The next elective topic, Pipe Flow and Aerodynamics, will go somewhat deeper into these two models of fluid friction and drag, with details on the differential methods that led researchers to expect something like the Darcy-Weisbach equation.

Reynolds Number and Turbulence

Assuming you don't do that elective, there is one important effect in the two experimental graphs we have examined in this supplementary lecture: The relationship of the Reynolds number to turbulence.

The horizontal axis in these graphs is the Reynolds number, plotted to a logarithmic scale because of the very wide range of values that Re can take.

By inspecting the graphs, it is evident that there are distinct bands with different behaviour: a low-Re band where the function is nearly linear, and a high-Re band where nonlinearities and quite complex behaviour are found.

The low-Re range is laminar, prior to the onset of turbulence. The high-Re range shows increasing intensity of turbulence. The Reynolds number, for this reason, is regarded as the basic measure of turbulence and will always appear in problems where turbulence is significant.

The Moody diagram (friction factor) also has a transitional region in which laminar flow begins to break up into turbulence. This region is not well understood; friction factor calculations in this zone are unpredictable, with large differences due to small effects of upstream conditions.

For a more complete introduction to turbulence, see that lecture in the next topic.

Mach Number

If you've read this far into a difficult supplementary chapter, you must be interested to know what the Mach number is for. Mach number similitude is essential to dimensional analysis and model scaling of problems with significant elastic effects, such as high speed gas flows, aircraft and propulsion.

These are covered to some extent in the next topic and in the advanced lecture *Isentropic 1-Dimensional High Speed Gas Flow* in the Complete Lectures section. These techniques are thermodynamic and extremely advanced: Such problems are at least 4-dimensional $MLT\Phi$ with temperature effects due to energy exchanges in compression, many of which are governed by the Mach number.

Wind Turbines and Propellers

This is an area of active research to optimise wind turbine blade design. Unfortunately it is fairly advanced, but is largely governed by dimensionless variables including the discharge ratio introduced in the previous topic. An accessible introduction is provided in Wind Turbine Blade Design³ in the complete lectures.

³ Schubel, P.J. and Crossley, R.J. 2012. *Wind Turbine Blade Design*. Energies ISSN 1996-1073
www.mdpi.com/journal/energies