

Lecture 19: Dimensional Analysis Continued

Area and Volume Ratios

In the last example a discharge ratio was devised more or less on the fly as a combination of dynamic (velocity) and geometric (area) ratios, to solve a particular problem. Different problems call for different combinations: a certain amount of creativity is possible so long as your formulations are dimensionally correct.

For example the *area ratio is equal to the square of the length ratio*, and is useful in problems of discharge or pressure force where actions apply to scaled areas. Similarly *the volume ratio is to the cube of the length ratio* and is important in mass, momentum and buoyant force calculations.

Area and volume ratios scale differently, so systems that rely on both (eg sail boats designed by sail area and hull displacement volume) produce inconsistencies between model and prototype designs – for example *ballast deficiency*: a model sailboat requires far more ballast than the prototype.

Time Ratios

Some problems involve scaling not only by the spatial scale but also by the *time* scale. For example, a model wave that is filmed in action and replayed at slow speed will appear to have a similar dynamic to a larger wave. This effect is familiar from old-fashioned scale model monster movies, where a man in a Godzilla suit is filmed in slow-mo attacking ships in a wave tank. The slowed-down wave action is more than superficially similar to large scale ocean waves.

Time scales provide a useful dynamic comparison of model to prototype for visualisation purposes. They are derived by consideration of velocity and length ratios:

In Reynolds systems, we have

$$V_r = v_r / L_r \quad (\text{Reynolds similitude})$$

But $V_r = L_r / t_r$ (Length ratio and time ratio related as per $V = L/t$)

So $L_r / t_r = v_r / L_r$

Or $t_r = L_r^2 / v_r$ **Time Ratio in Pipe or Submerged (Reynolds) Systems**

With model and prototype in the same fluid, $v_r = 1$, thus

$$t_r = L_r^2$$

ie a submerged model at $\frac{1}{2}$ the scale of a prototype will undergo a given action (ie traverse a given scaled length) in $\frac{1}{4}$ of the time the full scale prototype would take. To see a simulation of the full scale action, slow the film down to $L_r^2 = 25\%$ of real speed.

In Froude systems, we have

$$V_r = \sqrt{L_r} \quad (\text{Froude similitude})$$

But $V_r = L_r / t_r$

So $L_r / t_r = \sqrt{L_r}$

Or $t_r = L_r^{1/2}$ **Time Ratio in Surface (Froude) Systems**

ie a wave model at $\frac{1}{2}$ the scale of a prototype will undergo a given action in $0.5^{1/2} = 0.707$ of the time the full-scale prototype would take. To see a full-scale wave simulation, slow the film down to $L_r^2 = 70.7\%$ of real speed.

Euler Number: Pressure and Momentum Forces

Another important dimensional characteristic is the pressure-momentum force ratio, which arises in calculations of pressure forces arising from fluid accelerations. A convenient form is provided by the Euler Number, applicable to both surface and submerged or pipe flow:

$$Eu = \frac{P}{\rho V^2}$$

So for example a pressure force ratio can be formulated as $F_r = P_r A_r = \rho_r V_r^2 A_r = \rho_r V_r^2 L_r^2$

Where V_r is dependent on either Reynolds or Froude similitude, depending on the situation, and L_r is the length ratio.

For Reynolds similitude (pipe and submerged forces) the force ratio is $F_r = \rho_r V_r^2 L_r^2 = \rho_r L_r^2 / L_r^2 = \rho_r$

For Froude similitude (surface flow forces) the force ratio is $F_r = \rho_r V_r^2 L_r^2 = \rho_r L_r L_r^2 = \rho_r L_r^3$

Similarly power ratios can be calculated as Power = Force x Velocity: $Power_r = F_r V_r$

Example: The 2m model ship in the previous example is towed by a force of 7.2 N. What is the equivalent propulsion in the 120 m prototype? What power will that require?

The force ratio $F_r = P_r A_r = \rho_r V_r^2 A_r = \rho_r V_r^2 L_r^2$

Assuming the model is tested in the same density of water (eg salt or fresh) we have $\rho_r = 1$

The velocity ratio is given by Froude number, thus $V_r^2 = y_r = L_r = 1:60$

So $F_r = L_r^3 = (1:60)^3 = 4.6 \times 10^{-6} = F_m / F_p$

$$F_p = F_m / 4.6 \times 10^{-6} = 7.2 / 4.6 \times 10^{-6} = \underline{1.6 \times 10^6 \text{ N}}$$

We now have the prototype propulsion force and velocity (from previous question), so we can work out the power:

$$Power = FV = 1.6 \times 10^6 (5.4) = 8.4 \text{ MW}$$

Mach and Weber Numbers

The Mach number Ma is of most use in compressible flow situations and as such is outside of the scope of this course. See Granger (p382) for some general ideas about Ma .

Similarly the Weber number We is of most use in precision engineering of fluid surfaces where surface tension is comparable to inertia. Again, this is outside of the scope of this course. See Granger (p384) for some general ideas about We .

Similitude

The performance of a prototype (hydraulic device or machine) can in some circumstances be modelled to scale in lab conditions. For accurate modelling, the model and prototype must share certain types of *similitude*:

- **Geometric similitude** (similar shape): Consistent length ratios, including scaled surface roughness details, sometimes with exaggerated vertical to horizontal scales.
- **Dynamic similitude** (similar forces): Consistent ratios of dynamic pressures and forces at corresponding points on the model and prototype.
- **Kinematic similitude** (similar velocities): Consistent ratios of flow velocity at corresponding points: The overall shape of the streamline flow field is the same.

For exact similitude to exist, the Reynolds, Froude, Weber, Euler and Mach numbers calculated for the model and the prototype must be the same, ie all their ratios for model to prototype must equal 1.

Strictly speaking it is impossible to obtain similitude for all the dimensionless numbers at once, since fluid parameters such as viscosity, density and surface tension do not scale in the same way.

Approximate similitude is usually sufficient, and is found by considering the forces most relevant to a physical situation.

For example in ship design we find there is an inconsistency between the Reynolds and Froude numbers. Ideally we need to allow for skin friction (determined by Reynolds number) as well as wave forces (determined by Froude number) on the hull, but it is impossible to do so because the Froude and Reynolds numbers impose two entirely different relationships between velocity and length scale:

$$Re_r = \frac{D_r V_r}{\nu_r} = 1 \quad : \quad V_r = \frac{\nu_r}{D_r} \quad \text{Reynolds Similitude}$$

$$Fr_r = \frac{V_r}{\sqrt{y_r}} = 1 \quad : \quad V_r = \sqrt{y_r} = \sqrt{D_r} \quad \text{Froude Similitude}$$

It may be possible to engineer a model fluid with kinematic viscosity designed to balance the scales:

$$\text{Reynolds:} \quad D_r = \nu_r / V_r$$

$$\text{Froude:} \quad D_r = V_r^2$$

For both numbers to be consistent we need $\nu_r / V_r = V_r^2$ ie $\nu_r = V_r^3$

V_r can then be taken from the dominant Froude number (skin friction is much less than wave drag):

$$\nu_r = V_r^3 = y_r^{3/2} \text{ provides a possible model kinematic viscosity for a combined test.}$$

However this is often not technically feasible. In practice, skin friction and wave drag are modelled and measured in separate tests, and scaled separately to the prototype.

Similitude is not necessarily realistic other than in finding common patterns of behaviour. For example the force required to move a model plane at a velocity kinematically similar to that of a full scale plane is the same as the force on the full scale plane, assuming both are tested in air. This is because the kinematic model velocity is very large, much larger than a model plane would really fly.

The kinematic velocity provides an experimental benchmark between the model and the prototype, not a theoretical statement as to how either of them actually works.

General Procedure for Model: Prototype Scaling and Analysis

As a general rule, follow these steps for a scaling analysis:

1. Is your system a Reynolds (pipe or submerged) or Froude (surface flow) system?
2. Specify the model scale as a length ratio $L_r = L_m/L_p$
3. Calculate fluid density and viscosity ratios: $\nu_r = \nu_m/\nu_p$; $\rho_r = \rho_m/\rho_p$
4. Calculate relevant kinematic and dynamic ratios:

Reynolds

Derive and Calculate as needed:

Velocity Ratio	$V_r = \nu_r / L_r$
Time Ratio	$t_r = L_r^2 / \nu_r$
Pressure Ratio	$P_r = \rho_r \nu_r^2 / L_r^2$
Force Ratio	$F_r = \rho_r \nu_r^2$
Power Ratio	$P_r = \rho_r \nu_r^3 / L_r$

Froude

Derive and Calculate as needed:

Velocity Ratio	$V_r = \sqrt{L_r}$
Time Ratio	$t_r = \sqrt{L_r}$
Pressure Ratio	$P_r = \rho_r L_r$
Force Ratio	$F_r = \rho_r L_r^3$
Power Ratio	$P_r = \rho_r L_r^{7/2}$

1. With the ratios calculated, convert from model to prototype or vice versa as needed.

Unknown model fluid densities or viscosities require a little more thought, but these are less common problems in practice.

Dimensional analysis has proven to be of enormous benefit to the development of our understanding of real fluid behaviour. Fluid friction in particular is well suited to these methods, as almost all aspects of viscous and turbulent friction depend on Reynolds number. In Section 6 we will see that open channel hydraulics makes extensive use of the Froude number.

The remainder of this section deals with fluid friction theory and experiment, including theories of laminar and turbulent friction. This leads to applications in lubrication, pipe flow, simple pipe drainage systems, drag and lift, and concludes with an introduction to aerodynamics and propulsion theory.

2. Reading

Granger 1985 (Reading 9) Sections 7.6 to 7.9