

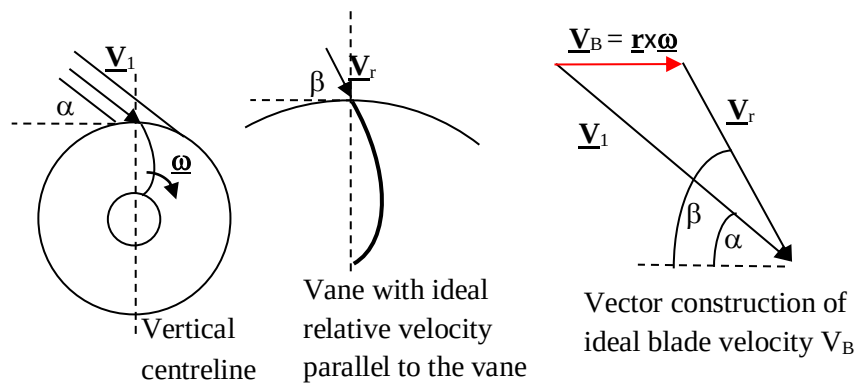
Supplementary Lecture 1: Vector Theory of Pumps and Turbines

This lecture is mathematically challenging and is not really essential to the standard design process for pumps and turbines. If you are interested in gaining an understanding of the momentum-reaction processes by which pumps and turbines operate, and are reasonably confident with vector algebra, start here. Otherwise we recommend you go on to Lecture 16.

The momentum theory is useful for an understanding of the underlying physics of turbomachinery and provides a guideline for detailed mechanical design, such as design of vane or blade angles, entry angles and inner/outer radii, all of which are optimised to the best efficiency point where the momentum theory actually works.

Radial or Centrifugal Turbomachines

A design rule for optimal performance of a radial turbine is to align the entrance flow velocity to the vane's initial deflection angle β within the vane's frame of reference, ie provide a smooth entry of fluid moving parallel to the vane:

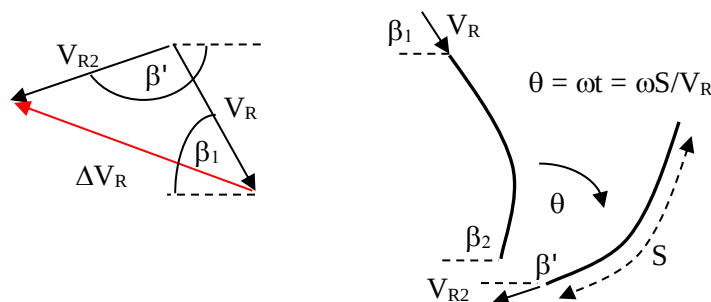


The parallel entry velocity in the vane's frame of reference ensures a smooth and direct entry flow with no shock or separation turbulence. In practice the turbine may operate at a range of speeds with varying efficiency, the optimum being close to the ideal relative velocity. α and β are thus design parameters of the entry flow: the angle of the inflow and of the vane to the horizontal as shown.

$$V_{Ry} = V_1 \sin \alpha = V_R \sin \beta : V_R = V_1 \sin \alpha / \sin \beta : V_{Rx} = V_R \cos \beta$$

$$V_{1x} = V_1 \cos \alpha : V_B = V_{1x} - V_{Rx} = V_1 \cos \alpha - V_R \cos \beta$$

An efficient blade deflection design β_1 to β_2 provides a maximum turning angle after allowance for the rotation of the blade:



The blade rotation angle β' is the exit deflection angle after the rotation of the vane in the time a slug of flow travels along the vane: $\beta' = \beta_2 + \omega t = \beta_2 + \omega S / V_R$ where S is the length of the vane.

The relative mass flowrate is given by:

$$\dot{m}_R = \rho \underline{V}_R \cdot \underline{A} \quad \text{where } \underline{A} \text{ is the flow area of } \underline{V}_1.$$

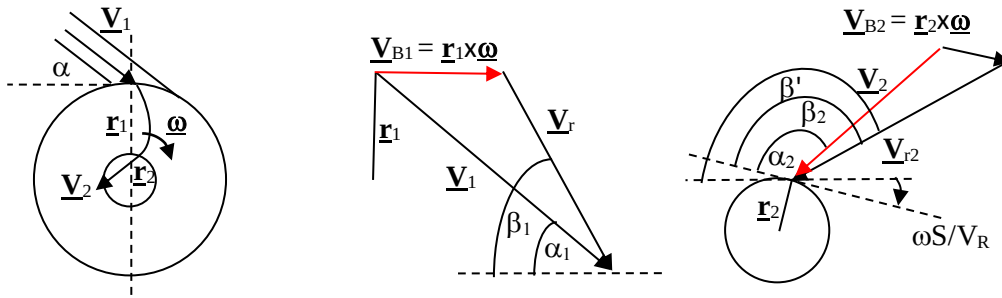
and $\dot{m} = \rho \underline{V}_1 \cdot \underline{A}$

so $\frac{\dot{m}_R}{\dot{m}} = \frac{\rho \underline{V}_R \cdot \underline{A}}{\rho \underline{V}_1 \cdot \underline{A}} = \frac{V_R \cos(\beta - \alpha)}{V}$

thus $\dot{m}_R = \dot{m} \frac{V_R}{V} \cos(\beta - \alpha)$

and $F = \dot{m}_R \Delta V_R$ as before.

A more detailed design model identifies two radii at which $\underline{F}$ applies, ie the vane radii at the angles β_1 and β_2 and converts these into a total Torque. The construction includes the additional angular displacement due to rotation of the vane in the time it takes a slug flow to pass through it:



The initial angular momentum is modified by an impulse provided by the vane, leaving the final angular momentum:

$$\text{Initial angular momentum} + \text{impulse} = \text{Final angular momentum}$$

$$dM r_1 V_1 \cos \alpha_1 + \text{torque} \times dt = dM r_2 V_2 \cos \alpha_2$$

but $dM = \dot{m} dt$. Substituting and eliminating,

$$\text{torque} = \dot{m} (r_2 V_2 \cos \alpha_2 - r_1 V_1 \cos \alpha_1)$$

This is the torque exerted by the vane on the fluid. The reaction torque exerted by fluid on vane is:

$$T = \dot{m} (r_1 V_1 \cos \alpha_1 - r_2 V_2 \cos \alpha_2)$$

All terms other than V_2 and α_2 are known from the inputs. V_2 and α_2 are calculated by the vector constructions:

$$\text{We have } V_R = V_1 \sin \alpha_1 / \sin \beta_1$$

V_{R2} has the same magnitude as V_{R1} (friction losses small, shared reference frame)

$$\text{Thus } V_{R2} = V_R (\cos \beta', -\sin \beta')$$

where $\beta' = \beta_2 + \omega t = \beta_2 + \omega S / V_R$ (β_2 rotated in the time t that V_R moves the length S of the vane)

So $\underline{V}_2 = (V_{2x}, V_{2y}) = (V_{R2x} + V_{B2x}, V_{R2y} + V_{B2y})$

$$= (V_R \cos \beta' + r_2 \omega \cos(\omega S / V_R), -V_R \sin \beta' - r_2 \omega \sin(\omega S / V_R))$$

$$V_2 = |\underline{V}_2|$$

And $\alpha_2 = \cos^{-1}(V_{2x} / V_2) - \omega S / V_R$

Particular care must be taken with the signs of vector components and the directional sense of angles calculated in these problems. From the torque, the turbine power can be calculated.

$$\text{Power} = \underline{T} \cdot \underline{\omega} = T\omega$$

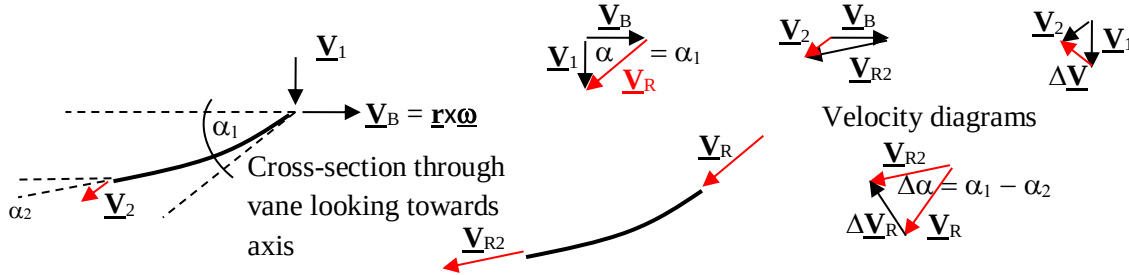
This spreadsheet solution for a radial turbine may be rerun in the appropriate electronic format or rewritten from the formulas shown: try changing some of the input variables (red font) to see what happens.

Discharge	m ³ /s	Q	0.8		
Inflow Pipe Diameter	m	D	0.375		
Inflow Pipe Area	m ²	$A = \pi D^2/4$	0.11044652		
Inflow Pipe Angle	rad	α_1	0.26179917	15	Degrees
Inflow Radius	m	r ₁	0.5		
Outflow Radius	m	r ₂	0.2		
Inflow Deflection	rad	β_1	1.1	63.03	Degrees
Outflow Deflection	rad	β_2	2.4	137.51	Degrees
Blade Length	m	$S = f(r_1 - r_2)$	0.322		
Inflow Velocity	m/s	$V_1 = Q/A$	7.243		
Mass Flow Rate	kg/s	$m = \rho V_1 A$	800		
Relative Velocity	m/s	$V_R = V_1 \sin \alpha_1 / \sin \beta_1$	2.104		
Blade Velocity	m/s	$V_B = V_1 \cos \alpha_1 - V_R \cos \beta_1$	6.042		
Angular velocity	rads/s	$\omega = V_B / r$	12.085	115.4	RPM
Rotation in slug flow time	rad	$\theta = \omega S / V_R$	1.851	106.06	Degrees
True Relative Outflow Angle	rad	$\beta' = \beta_2 + \theta$	4.251	243.57	Degrees
$\underline{V}_2$	m/s	(V _{2x} , V _{2y})	-1.605	-0.44	
V ₂	m/s	$ \underline{V}_2 $	1.664		
Outflow angle	rad	$\alpha_2 = \cos^{-1}(V_{2x}/V_2) - \theta$	1.024	58.64	Degrees
Active Torque	N.m	$T = m \cdot (r_1 V_1 \cos \alpha_1 - r_2 V_2 \cos \alpha_2)$	2,660		
Torque Power	W	$P = T\omega$	32,146		
Linear Power	W	$P = m \cdot V_1^2$	41,973		
Transfer Efficiency			77%		

Axial Turbines

In a gas turbine a heated or burned exhaust gas at velocity V_1 is passed in an axial direction through a fan or series of fans, usually in a closed armature, turning a generator. This is much like a fan or propeller in reverse. Portable gas generators are a convenient size for a semi-trailer and consist of a gas burning compressor venting into a flow chamber where the fan system is mounted.

The construction for an axial momentum system is more complex because of the variation in blade velocity along the length of a vane. One solution is to consider a typical radius at about 2/3 the length of a single vane:



As with the ideal radial flow the relative velocity due to rotation of the vane ideally aligns with the vane's initial deflection angle, ie $\alpha = \alpha_1$ as this gives minimum shock and drag characteristics. Assuming it does so provides an optimal performance model.

$$V_R \sin \alpha_1 = V_1 \quad V_R \cos \alpha_1 = V_B \quad \underline{V}_R = \underline{V}_1 - \underline{V}_B = (-V_B, -V_1) \quad \underline{V}_{R2} = (-V_R \cos \alpha_2, -V_R \sin \alpha_2)$$

The relative velocity turns through an angle $\Delta \alpha$ giving the relative velocity change $\Delta \underline{V}_R$:

$$\Delta \underline{V}_R = \underline{V}_{R2} - \underline{V}_R = (-V_R \cos \alpha_2 + V_R \cos \alpha_1, -V_R \sin \alpha_2 + V_R \sin \alpha_1) = V_R (\cos \alpha_1 - \cos \alpha_2, \sin \alpha_1 - \sin \alpha_2)$$

The horizontal acceleration derives from the resultant active force on the fluid. The corresponding reaction force on the blade is opposite to the active force:

$$\underline{F} = -\dot{m}_R \Delta \underline{V}_R$$

and $F_x = -\dot{m}_R \Delta V_x$

The relative mass flowrate is given by:

$$\dot{m}_R = \rho \underline{V}_R \cdot \underline{A} \quad \text{where } \underline{A} \text{ is the flow area of } \underline{V}_1.$$

and $\dot{m} = \rho \underline{V}_1 \cdot \underline{A}$

so
$$\frac{\dot{m}_R}{\dot{m}} = \frac{\rho \underline{V}_R \cdot \underline{A}}{\rho \underline{V}_1 \cdot \underline{A}} = \frac{V_R \cos \alpha}{V}$$

thus
$$\dot{m}_R = \dot{m} \frac{V_R}{V_1} \cos \alpha$$

The mass flowrate per vane is typically taken as the total flowrate divided by the number of vanes.

The vane torque is the reaction force acting over the average radial distance: $\underline{T} = \underline{r} \times \underline{F}$. From this the power of the vane is $P = \underline{T} \cdot \underline{\omega}$. For the total torque or power of the turbine, multiply by the number of vanes.

Again the efficiency can be estimated by comparing this to the power of the linear inflow: $P = \dot{m} V_1^2$

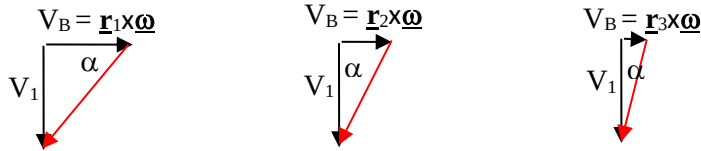
The efficiency of the transfer is thus
$$\eta = \frac{n T \omega}{\dot{m} V_1^2}$$

Axial Vane Angles and Profiles

In order to align the vane velocity with the inflow velocity at each radius, the vane deflection angle must vary along the length of the vane: This produces the classic turned shape of propellers and efficient fan blades:

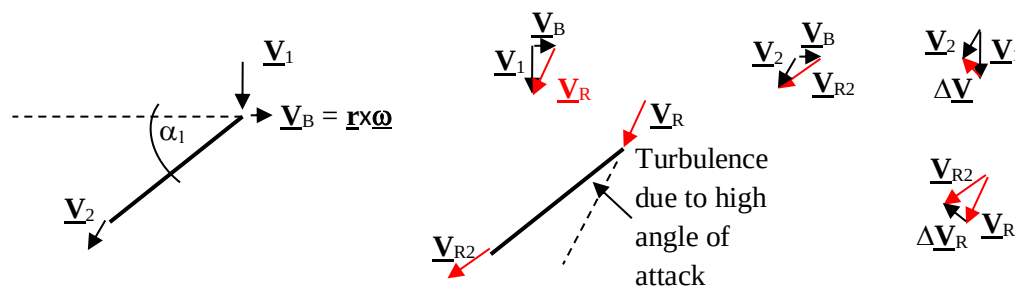
$$\underline{V}_B = \underline{r} \times \underline{\omega}$$

$\underline{V}_B$ is smaller with decreasing radius. V_1 is constant, giving a varying deflection angle:



Deflection angle varies with radius: the classic turned profile of an axial vane.

At any radius the deflection angle changes from α_1 to α_2 . An efficient design of α_1 to α_2 is required to achieve optimal performance where the relative velocity aligns to the vane. A *flat vane* ($\alpha_1 = \alpha_2$) must have a high relative incidence angle to generate a usable velocity change. The high angle creates turbulence, while the smaller reaction produces low operating speeds and low torque:



Flat axial vane with poor operating speed and limited active force.

Propellers are designed for a very high flow velocity change, sufficient to induce thrust and lift, and use an aerodynamic profile, effectively a specialised wing. Propellers are examined in further detail in Section 5.4, Propulsion.

Reading

Fox and McDonald 1992 (Reading 8) Section 11.1 to 11.3

Schubel & Crossley, 2012. Wind Turbine Design (Compete Lectures)