

Lecture 15: Conservation of Momentum

Many technical problems require not only the continuity and energy equations for solution, but also depend on the conservation of momentum. Examples include the reaction forces of fluid accelerations on the walls of a conduit, and the dynamic pressures of pump or turbine vanes.

Conservation of Momentum: The Force-Reaction Law

An element of mass has momentum:

$$d\vec{p} = \vec{v}dm = \vec{v}\rho dV$$

The rate of change of momentum is equal to the net resultant force acting on a system (Newton's 2nd Law, Force = mass x acceleration):

$$\sum \underline{F} = \frac{D}{Dt} \oint_s \underline{V} \rho \, dV$$

Similarly the rate change of angular momentum is equal to the net resultant moment:

$$\sum \underline{M} = \frac{D}{Dt} \oint_s \underline{r} \times \underline{V} \rho \, dV$$

Taking the force relation first, and noting that momentum per unit mass is just velocity,

$$\underline{\underline{\Sigma \mathbf{F}}} = \frac{\partial(\rho \underline{\underline{\mathbf{V}}})}{\partial t} + \nabla \cdot \rho \underline{\underline{\mathbf{V} \mathbf{V}}} \quad \text{Newton's Second Law for Fluids}$$

Local Convective

For steady flow the local acceleration is zero. In uniform flow with the flux form of divergence,

$$\sum \underline{\mathbf{F}} = \nabla \cdot (\rho \underline{\mathbf{V}} \underline{\mathbf{V}}) = \Delta \rho \underline{\mathbf{V}} \underline{\mathbf{V}} \cdot \underline{\mathbf{A}}$$

Again, ρVA is the mass flowrate $\dot{m}$, thus:

$$\sum \underline{F} = \dot{m} \Delta \underline{V} \quad \text{Momentum Reaction Law (steady)}$$

Resultant of active forces Vector Change in velocity

Mass flowrate

Similarly for angular momentum,

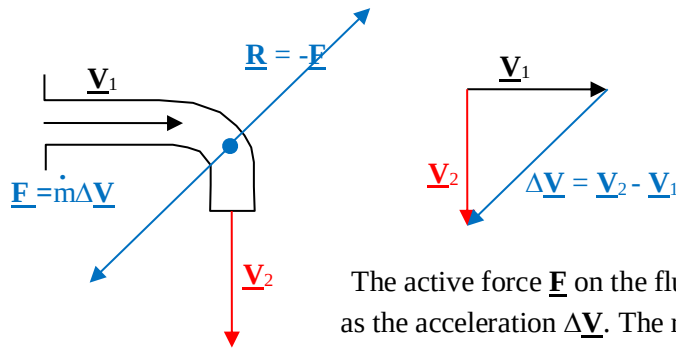
$$\sum \underline{M} = \dot{m} \Delta(\underline{r} \times \underline{V}) \quad \text{Moment of Momentum (Steady)}$$

Resultant of active moments Vector Change in angular velocity

Mass flowrate

The vector nature of $\Delta \mathbf{V}$ can be clarified by vector diagrams:

For a bend:



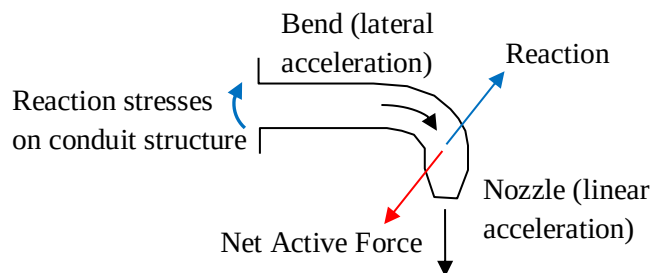
The active force $\underline{F}$ on the fluid is in the same direction as the acceleration $\Delta\underline{V}$. The reaction force $\underline{R}$ of the fluid reacting on the structure is opposite to $\underline{F}$.

Similarly for velocity change in a straight line (ie a contraction), the $\Delta\underline{V}$ vector is constructed by connecting $\underline{V}_1$ and $\underline{V}_2$ head to head:



$\Delta\underline{V}$ is a *relative* velocity change, or a *convective acceleration* since it occurs even in steady flow.

Combining a bend with a nozzle (not at all uncommon) we use both:



The active force is developed by a distribution of stresses, both normal (pressure) and shear (viscous friction) from the supporting structure on the fluid. The position of the resultant force can be found by consideration of moment balance as per hydrostatic pressure loads on structures (see examples in tutorials and assignment 3). As with hydrostatic loads, these dynamic loads must be supported by structural mounts, for example the massive blocks that support any bend in a high speed pipeline.

Non-uniform flow profiles

As with energy conservation, a simple form of divergence is found by assuming uniform velocity distributions. Other distributions require a momentum correction factor, β :

$$\beta = \frac{\int V^2 dA}{\bar{V}^2} \text{ where } \bar{V} \text{ is the mean flow velocity.}$$

then $\sum \underline{F} = \beta \dot{m} \Delta\underline{V}$ is the modified momentum reaction law.

$\beta = 1$ (turbulent, near-uniform)

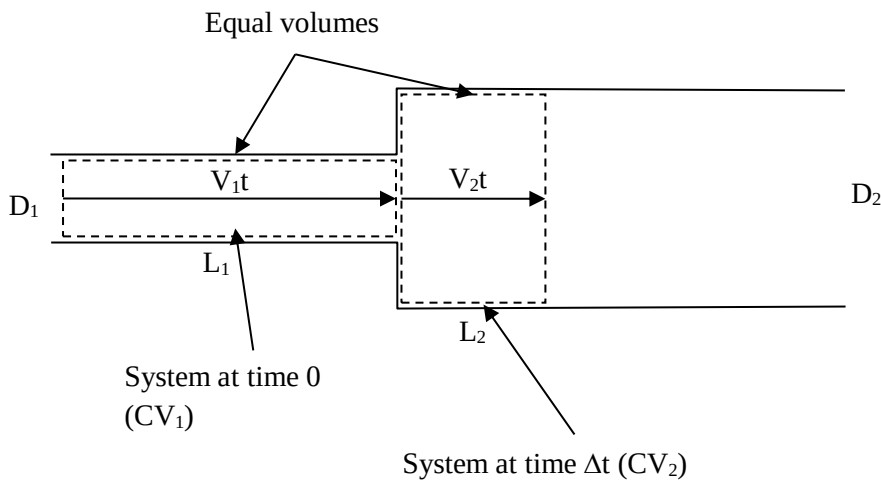
$\beta = 1.33$ (laminar)

Conservation of Momentum: The Easy derivation

Any net change in momentum must be balanced by a corresponding reaction force as described by Newton's second law. This is usually worked out for local, transient forces such as a collision or acceleration of a fixed mass:

$$F = \frac{d(mV)}{dt} = \frac{dp}{dt}$$

We can consider a transient incompressible reaction as per the simple continuity derivation. A system of mass m moves between two control volumes CV_1 and CV_2 in a short reaction time Δt . The mass is constant, but the velocity and hence momentum must change due to continuity. The momentum change divided by the reaction time must equal the reaction force:



The reaction time can be worked out kinematically:

$$\Delta t = \frac{L_1}{V_1} = \frac{L_2}{V_2} \quad (\text{as distance} = \text{velocity} \times \text{time})$$

But $L_1 = \frac{\forall}{A_1}$ and $L_2 = \frac{\forall}{A_2}$

so $\Delta t = \frac{\forall}{V_1 A_1} = \frac{\forall}{V_2 A_2} = \frac{\forall}{Q}$

Thus $\frac{dp}{dt} \sim \frac{\Delta p}{\Delta t} = \frac{m(V_2 - V_1)}{\frac{\forall}{Q}} = \frac{mQ(V_2 - V_1)}{\forall}$

But $\frac{m}{\forall} = \rho$

So $F = \frac{dp}{dt} = \rho Q(V_2 - V_1) = \rho Q \Delta V$

A fluid mass in a state of flow is not a body of fixed mass and is not transient. A flow is a continuous and extended stream of mass, which in the region of a bend or contraction undergoes a continuous change in velocity. For a steady change in velocity there must be a steady force.

This suggests that there are two possible types of fluid acceleration:

1. An acceleration may be produced by an unsteady flow where flowrates and hence velocities at any point in the flow are increasing. This *local* acceleration (it occurs at every location in the flow) is described by the conventional time derivative of velocity, ie the acceleration at any location. *Local forces can be very large, as the entire fluid body is accelerated as in system start-up or the water hammer phenomenon.*
2. In steady flow, an acceleration occurs wherever a change in pipe diameter or direction occurs. This *convective* acceleration (it arises from mass movement through a non-uniform, spatially changing flow field) is described by a chain-rule derivative of the velocity:

$$\frac{dV}{dt} = \frac{dV}{dx} \frac{dx}{dt} = \frac{dV}{dx} V \quad \text{Convective Acceleration}$$

Here the spatial derivative dx shows that the fluid is moving through the flow field and the velocity is changing as it does so. So for steady convective acceleration we have:

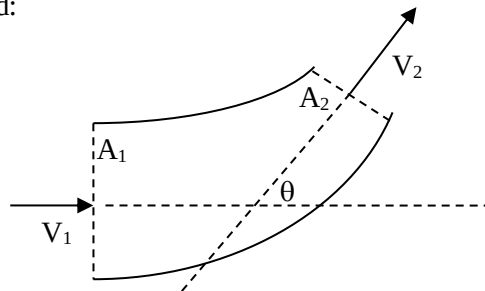
$$F = \rho \nabla \frac{dV}{dt} = \rho \nabla \frac{dV}{dx} \frac{dx}{dt} = \rho \frac{\nabla}{dt} dV = \rho Q dV = \dot{m} \Delta V$$

There is in fact no easy derivation of the continuity of momentum as applied to fluids. The best way to understand it is to experience it. If you can, go outside and turn on the hose. Where's the nozzle attachment?

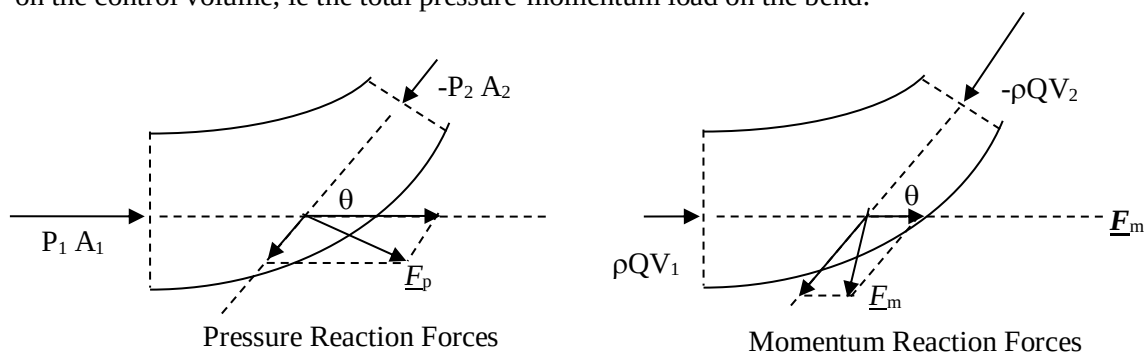
Combining Pressure Forces and Momentum-Reaction Forces

In practice a designer needs to consider pressure forces according to Bernoulli's theorem as well as momentum reaction forces. For a reducing bend, the solution of forces acting on the bend requires mass, energy and momentum conservation. The situation can be depicted with a force diagram:

Example: a reducing bend:



We can draw force diagrams for pressure and momentum. It's easier to draw them separately, and then add them together. It is also useful to adopt a sign convention that indicates the direction of active forces on the control volume, ie the total pressure-momentum load on the bend:



Given a flowrate Q , pipe geometry A_1 , A_2 and θ , and an initial pressure P_1 , we can solve for:

$$V_1 = \frac{Q}{A_1} \quad V_2 = \frac{Q}{A_2} \quad P_1 + \frac{\rho V_1^2}{2} = P_2 + \frac{\rho V_2^2}{2} \quad \dot{m} = \rho Q$$

Pressure and momentum forces can now be evaluated by multiplying into unit vectors and adding or subtracting as per vector rules. The unit vectors aligned to the areas are $(1, 0)$ and $(\cos \theta, \sin \theta)$, so:

Pressure Forces:

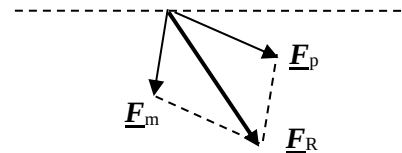
$$\underline{F}_p = P_1 \underline{A}_1 - P_2 \underline{A}_2 = P_1 A_1 (1, 0) - P_2 A_2 (\cos \theta, \sin \theta) = (P_1 A_1 - P_2 A_2 \cos \theta, -P_2 A_2 \sin \theta) = (F_{px}, F_{py})$$

Reaction Forces:

$$\underline{F}_m = \rho Q (\underline{V}_1 - \underline{V}_2) = \rho Q (V_1 (1, 0) - V_2 (\cos \theta, \sin \theta)) = \rho Q (V_1 - V_2 \cos \theta, -V_2 \sin \theta) = (F_{mx}, F_{my})$$

Resultant Force:

$$\underline{F}_R = (F_{px} + F_{mx}, F_{py} + F_{my})$$

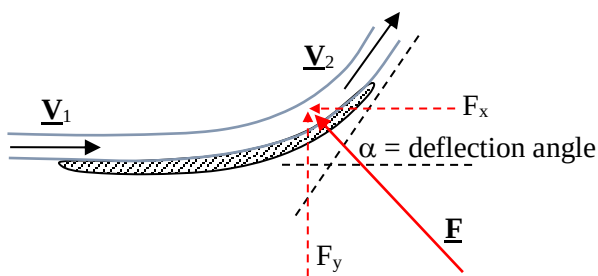


This is the force on the control volume, a combined pressure and momentum-induced load on the structure that supports this bend in the pipeline. Adequate structural restraint must be provided.

Applications of Momentum Conservation: Vanes and Deflectors

A vane is a moving element of a momentum transfer device such as a pump, turbine, propeller or wind power generator. The deflection angle *and* motion of the vane must both be taken into account to determine the force that a flow exerts on an individual vane.

Starting with just the deflection angle (ie assume a stationary vane) we resolve the active force into horizontal and vertical components:



Assume $|\underline{V}_1| = |\underline{V}_2| = V$

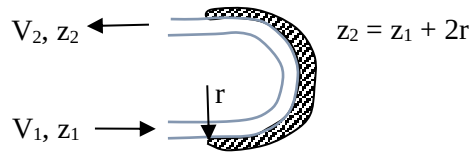
$$F_x = \dot{m} \Delta V_x = \dot{m} (V \cos \alpha - V) = \dot{m} V (\cos \alpha - 1)$$

$$F_y = \dot{m} \Delta V_y = \dot{m} (V \sin \alpha - 0) = \dot{m} V \sin \alpha$$

This assumes that the magnitude of velocity is unchanged, ie there are no losses, significant pressure differences or diversions (expansions/contractions) through the deflection: This is reasonable, considering the brief duration of the flow's passage through the vane.

In fact the velocity in the vane is a factor in an energy exchange and must be handled via Bernoulli's equation:

For example a vane turns a flow 180° through a radius r :

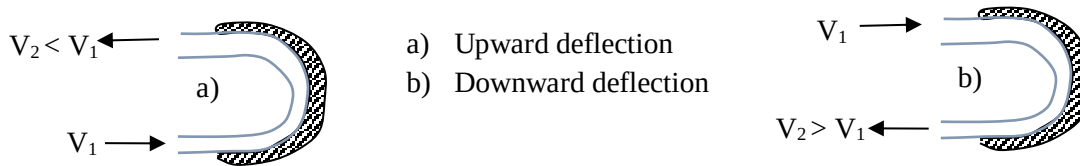


Taking a free surface streamline where $P = 0$, Bernoulli's equation is

$$\frac{V_1^2}{2g} + z_1 = \frac{V_2^2}{2g} + z_2$$

or
$$\frac{V_1^2}{2g} - \frac{V_2^2}{2g} = z_2 - z_1$$

that is the flow velocity head changes in proportion to the (small) change in elevation. Upward deflecting vanes tend to slow slightly (with resulting deeper flow) while downward deflecting vanes tend to speed up. The effect is small, but suggests that downward-pointing vanes are marginally more effective:



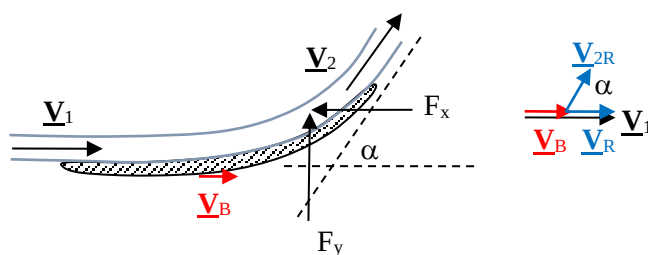
Moving Vanes and Deflectors

The movement of the vane reduces the relative velocities of inflow and outflow in the direction of motion. This reduces the efficiency of the vane but is a necessary condition for extracting useful work from it – the vane must move in order to extract energy from the flow.

Not only velocity but all flow rates including mass flow rates are affected by the vane's movement: If V_B is the velocity of the vane, the relative inflow velocity and relative mass flow rate are:

$$\underline{V}_R = \underline{V}_I - \underline{V}_B \quad ; \quad V_R = |\underline{V}_R|$$

$$\dot{m}_R = \rho A V_R = \dot{m} \frac{V_R}{V_I}$$



Both horizontal and vertical velocities are affected by the moving reference frame: The relative mass flow rate is constant, hence V_R and V_{2R} are equal in magnitude

Forces, as a source of energy, are defined relative to the reference frame in which they act: Force is exerted on the vane. Hence we must resolve forces within the vane's reference frame.

$$F_x = \dot{m}_R \Delta V_{xR} = \dot{m}_R V_R (\cos \alpha - 1)$$

$$F_y = \dot{m}_R \Delta V_{yR} = \dot{m}_R V_R \sin \alpha$$

The *power* or time rate of energy output of the vane is defined in the stationary reference frame. The resultant force $\underline{F}$ produces energy at the velocity of the reference frame in which it applies, ie $\underline{V}_B$:

$$P = \underline{F} \cdot \underline{V}_B = F_x V_B \text{ J s}^{-1} \text{ or Watts}$$

The relative mass flowrate assumes that V_1 and V_B are parallel: This *no-shock* condition provides a smooth transition from the entry flow to the vane, maximising the vane's efficiency:

$$\begin{aligned} \dot{m}_R &= \rho A V_R = \rho A (V_1 - V_B) \\ &= \dot{m} - \rho A V_B \\ &= \dot{m} - \dot{m} \frac{V_B}{V_1} \\ &= \dot{m} \left(\frac{V_1 - V_B}{V_1} \right) \\ &= \dot{m} \left(\frac{V_R}{V_1} \right) \end{aligned}$$

Rotating Vanes and Deflectors

The most common configuration for any kind of turbine or pump is circular, ie a *rotating* impeller with multiple fixed vanes around the circumference. Vane groups may be aligned in either of two directions relative to the flow: axial or radial.

Radial systems (sometimes called centrifugal) provide large head differences, while axial systems provide large flows with small head difference and are often used in gas pumps eg fans. Intermediate designs with cross-flow (diverging or converging) components, called mixed-flow boosters, are used in the middle range, ie to provide moderate head and discharge in liquids or gases.

To a certain extent pumps and turbines can be considered as equivalent devices operating in the reverse of each other. A pump would be reasonably efficient as a turbine if it was operated in reverse at a similar flowrate. The design process for pumps is therefore very similar to turbines.

The momentum theory of the preceding section is useful to gain an insight into the operation of a pump or turbine: velocity changes can be analysed quite closely, and the reaction forces calculated are similar in practice to the most efficient real performance. This point occurs at one particular speed.

However in practice pumps and turbines, or *turbomachines* as they are collectively called, operate over a range of speeds where turbulence and friction appreciably reduce efficiency. These effects are very difficult to model theoretically. Pump or turbine systems are designed and modelled by capacity and efficiency charts based on performance testing (see Turbomachinery).

Pump Types

There are two main types of pumps, with several subdivisions:

- Positive displacement pumps
 - Lift
 - Squeeze
- Roto-dynamic pumps
 - Radial or Centrifugal Flow
 - Axial Flow
 - Mixed Flow

Positive displacement pumps essentially involve lifting (in the sense of carrying) a fluid to a higher level, or squeezing it to push it upwards or to higher pressure. Examples are hand or foot pumps, Archimedean screws, and simple rotary bucket lift devices. More sophisticated lift devices include the rotary pump, in which one or two rotating cams move fluid through a fitted chamber and out the other side. The heart is a biological example of a positive displacement squeeze pump.

Roto-dynamic pumps use modern machinery (since the steam engine) to rapidly rotate a system of vanes which may be aligned radially or axially around the axis of rotation, or a mixture of radial and axial vanes. The following lecture covers the vector theory of radial and axial turbines. Similar constructions apply to pumps. Performance test modelling and design based on the theory of turbomachinery is provided in the subsequent lecture.

Reading

Potter and Wiggert 1991 (Reading 6) Section 4.5 to 4.7