

## ***Lecture 13: Applications of Continuity and Energy Conservation***

Application of the Continuity and Bernoulli equations should be systematic. A suggested procedure is as follows:

- 1) Draw a sketch of the system, labelling all cross-sections of the system under consideration.
- 2) Select a datum plane or horizontal line for vertical references. Use the lowest point in the system to avoid negative values.
- 3) Write the Bernoulli and continuity equations for each selected cross section, working from left to right in the equations as you work from upstream to downstream.
- 4) Use consistent units, preferably in metres of head for all pressure, velocity and elevation terms. Use gauge or absolute pressures consistently.
- 5) Evaluate the energy at a section upstream (or downstream) by inserting known values, then work downstream (or upstream) by using continuity to calculate unknown velocities and Bernoulli to calculate unknown pressures. (Your starting information may be at either the upstream or the downstream end, and you can solve in either direction)
- 6) Add energy input by pumps, and subtract energy due to losses or extraction by turbines.
- 7) The upstream energy, plus inputs, minus losses and extraction, is equal to the downstream energy. Equate this to the sum of pressure head, velocity head and elevation head at either upstream or downstream sections.

### ***Applications 1: Low Speed Gas Dynamics***

The temperature and density of a gas are additional variables which must be taken into account when using the Bernoulli equation for gas flow. It is important to express all pressures and temperatures in absolute terms, ie temperatures in Kelvin and pressures in Pa gauge + atmospheric, since these measurement systems are derived from the kinetic-molecular behaviour of gases.

For gas flow problems where significant compression occurs it is essential to use Bernoulli's equation in the form of pressure rather than head:

$$\frac{\rho_1 V_1^2}{2} + \rho_1 g z_1 + P_1 + \text{inputs} - \text{outputs} - \text{losses} = \frac{\rho_2 V_2^2}{2} + \rho_2 g z_2 + P_2$$

The full thermodynamic treatment of high speed gas flow is beyond the scope of this course, however for low speed flows ( $< \sim 100\text{m/s}$ ) two additional sources of information can be used:

1. The Ideal Gas Law:  $P = \rho R T$

Where R is a specific gas constant, equal to the universal gas constant divided by the gas molecular mass:

$$R = \frac{8.31}{M}$$

eg R (air,  $M = 29\text{g/mole}$ ) =  $8.31/(0.029) = 287 \text{ J/Kg.K}$

2. A Process or Constitutive Law:  $\frac{P}{\rho^n} = \text{constant}$

Where n depends on the process:

Isothermal (Constant Temperature):  $n = 1: P/\rho = k$

|                                 |                                                     |
|---------------------------------|-----------------------------------------------------|
| Isobaric (Constant Pressure):   | $n = 0: P = k$                                      |
| Isentropic (or Adiabatic):      | $n = \gamma: P/\rho^\gamma = k$                     |
| Polytropic (Natural Processes): | $n = \text{other values determined by experiment.}$ |
| Incompressible                  | $\rho = \text{constant}$                            |

These additional gas properties, in particular the ratio of heat capacities  $\gamma = C_p/C_v$ , are discussed in Section 1, Fluid Properties.

A brief introduction to the thermodynamic treatment of isentropic one-dimensional flow is provided in the Supplementary Tutorial, Isentropic Compressible Flow

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Example: A pipe 150 mm in diameter contracts down to 100 mm diameter. If air at initial density  $1.2 \text{ kg/m}^3$  and  $20^\circ\text{C}$  flows at  $20 \text{ m/s}$  in the upstream section, what is its velocity in the downstream? Assume a) isothermal compression, b) isentropic compression,  $\gamma_{\text{air}} = 1.4$ ,  $R_{\text{air}} = 287 \text{ J/kgK}$

a) For compressible fluids the mass flowrate is a constant:  $\rho_1 V_1 A_1 = \rho_2 V_2 A_2 = 0.424 \text{ kg/s}$

For isothermal (constant temperature) conditions the process  $P/\rho = k$  gives  $P_1/\rho_1 = P_2/\rho_2$

Using Bernoulli's equation with no inputs, outputs, losses or elevation change we have:

$$\frac{\rho_1 V_1^2}{2} + P_1 = \frac{\rho_2 V_2^2}{2} + P_2$$

The pressure at  $20^\circ\text{C}$  ( $=20+273 = 293 \text{ K}$ ) and  $1.2 \text{ kg/m}^3$  is

$$P = \rho RT = 1.2(287)(293) = 100,909 \text{ Pa } (\sim 1 \text{ Atmosphere})$$

We can substitute the continuity and process equations into Bernoulli's equation:

$$\begin{aligned} \frac{\rho_1 V_1^2}{2} + P_1 &= \frac{(\rho_2 V_2) V_2}{2} + P_1 \frac{\rho_2}{\rho_1} \\ &= \left( \frac{\rho_1 V_1 A_1}{A_2} \right) \frac{V_2}{2} + P_1 \frac{V_1 A_1}{V_2 A_2} \end{aligned}$$

Multiplying through by  $V_2$  gives

$$\left( \frac{\rho_1 V_1^2}{2} + P_1 \right) V_2 = \left( \frac{\rho_1 V_1 A_1}{A_2} \right) \frac{V_2^2}{2} + P_1 \frac{V_1 A_1}{A_2}$$

This is a quadratic equation with two solutions,  $V_2 = 45.4$  or  $3701 \text{ m/s}$ .

The lower value of  $45.4 \text{ m/s}$  is a physically realistic solution.

The density in the smaller diameter duct is found by the mass flowrate:

$$\rho_2 V_2 A_2 = 0.424 \text{ kg/s} \quad : \quad \rho_2 = 0.424(45.4(\pi 0.1^2/4)) = 1.189 \text{ kg/m}^3$$

Which for the same temperature gives a pressure  $P_2 = \rho RT = 1.189(287)(293) = 99984.2 \text{ Pa}$

We can check this with Bernoulli's equation:

$$1.2(20)^2/2 + 100909 = 101149 = 1.189(45.4)^2/2 + 99984.2 \quad \text{OK}$$

Note: Isothermal conditions are unusual and must be specifically engineered in place. An isentropic flow with  $P/\rho^\gamma = k$  is more realistic, but more complex to model with both density *and* temperature changes:

b) For adiabatic conditions the process is  $P/\rho^\gamma = k$ :

$$\text{Thus } P/\rho^\gamma = 100,909/1.2^{1.4} = 78,176$$

Which gives  $P = 78,176\rho^\gamma$  as a variable in Bernoulli's equation

We can use the compressible continuity equation for  $V$ :

$$\rho VA = \dot{m} = 0.424$$

Thus  $V = \dot{m}/\rho A$  ie  $V^2 = (0.424/\rho A)^2$  provides another variable.

Bernoulli's equation thus reduces to density changes:

$$\frac{1.2 \times 20^2}{2} + 100,909 = \frac{\rho_2 \left( \frac{0.424}{\rho_2 A_2} \right)^2}{2} + 78,176 \rho_2^\gamma = \frac{1457}{\rho_2} + 78,176 \rho_2^\gamma$$

Solving the LHS,

$$101,149 = \frac{1457}{\rho_2} + 78,176 \rho_2^\gamma$$

This can be solved by trial and error:  $\rho_2 = 1.192 \text{ kg/m}^3$

Which gives  $V_2 = 0.424/(1.192 \cdot A_2) = \underline{45.3 \text{ m/s}}$

Pressure *and* temperature change:

$P_2 = 78,176(1.192)^{1.4} = 99968 \text{ Pa}$  (or by solving Bernoulli's equation for  $P_2$  with  $V_2$  known)

$T_2 = P_2/\rho_2 R = 99968/(1.192 \cdot 287) = 292 \text{ K} = 17^\circ \text{C}$ .

NOTE THAT FOR MODERATE ACCELERATIONS, THE ISOTHERMAL AND ISENTROPIC SOLUTIONS ARE BOTH NEARLY IDENTICAL TO AN ASSUMPTION OF ZERO COMPRESSION: THIS IS A VERY USEFUL SHORTCUT

Incompressible Continuity:  $V_2 = V_1 A_1/A_2 = 45 \text{ m/s}$

Pressure and temperature changes become more extreme with increasing velocity. As the velocity approaches or exceeds sonic velocity  $c = \sqrt{\gamma RT}$  this method becomes unstable and breaks down (the polynomial has no real-number solution). A specific combination of pressures upstream and downstream are required to accelerate a gas past this speed. Care should be exercised for  $V > 0.3c$ .

Example: Compressed air at  $9^\circ \text{C}$  escapes from a tank at 207 kPa absolute into atmosphere at 101,300 Pa. What is the velocity of the leaking gas at the outlet assuming an isentropic process?

The velocity and pressure inside the tank are known:  $V_1 = 0$ ,  $P_1 = 207 \text{ kPa}$  absolute.

The density inside the tank is  $\rho = P/RT = 207 \times 10^3 / (287(9+273)) = 2.56 \text{ kg/m}^3$

The isentropic process is  $P/\rho^\gamma = k = 207 \times 10^3 / 2.56^{1.4} = 55589$ . The leaking gas emerges with its own distinct temperature and hence density, but its pressure drops immediately to atmospheric pressure:

$$P_2/\rho_2^\gamma = 101300/\rho_2^\gamma = 55589 : \rho_2 = (101300/55589)^{1/\gamma} = 1.53 \text{ kg/m}^3$$

Combining terms in Bernoulli's equation:

$$0 + 207 \times 10^3 = \frac{1.53 V_2^2}{2} + 101300 : V_2 = \sqrt{\frac{2}{1.53} (207 \times 10^3 - 101300)} = \underline{371.7 \text{ m/s}}$$

This is greater than the speed of sound, a condition which must be specifically engineered in place. This indicates that an isentropic solution is unrealistic at this pressure, so losses are significant.

Nevertheless an estimate of the escape temp is  $T = P/\rho R = 101300/(287 \cdot 1.53) = 231 \text{ K} = \underline{-42^\circ \text{C}}$ .

Leaking gas is hazardous for two less than obvious reasons; high velocity and low temperature, as well as the more familiar risks of toxicity and flammability.

## Applications 2: Total Energy and Hydraulic Grade Lines

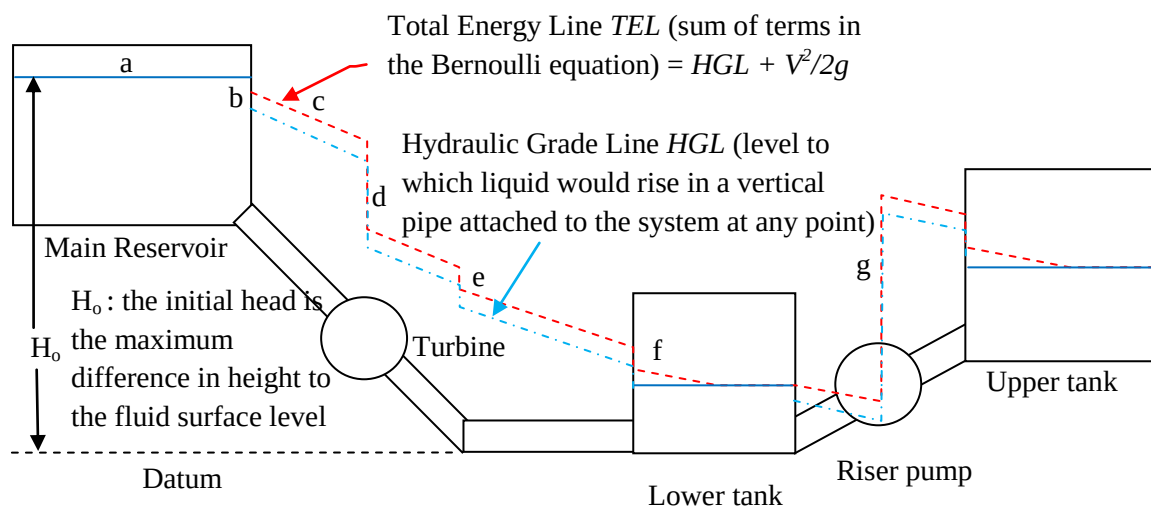
Bernoulli's equation provides a breakdown of fluid energy into its three principle components: kinetic (velocity head  $V^2/2g$ ), potential (free surface head  $z$ ) and elastic (pressure head  $P/\rho g$ ).

Energy inputs (pumps), outputs (turbines) and losses (friction etc) are accounted for in the same terms, ie head and headloss in units of vertical height, metres relative to some datum such as mean sea level. The formulation in terms of head is usually highlighted by the use of  $H$  rather than  $z$  for elevation.

The *Total Energy Line (TEL)* can thus be drawn to the same vertical scale as the system itself, including rises and falls in the landscape, structures and devices etc at different elevations.

The *TEL* includes the velocity head. If this term is excluded, ie only the potential terms  $P/\rho g + z$  are included, we obtain the *Hydraulic Grade Line (HGL)*, which is the level to which water would rise in a vertical pipe attached to the system:  $TEL = HGL + V^2/2g$

It is useful to sketch an energy grade line with any pipeline system. This helps to clarify the sign of energy inputs and outputs and on which side of the equation to include them:



- a) Main reservoir free surface level
- b) Outlet loss (contraction)
- c) Pipe loss (friction)
- d) Turbine Head (energy extraction)
- e) Bend loss (turbulence)
- f) Inlet loss (expansion, turbulence)
- g) Pump Head (energy injection)

Working from upstream to downstream or vice versa, changes in the total kinetic and potential energy due to all forms of inputs, outputs and losses can be formulated as functions of head, often proportional to the velocity head.

The details of loss terms in particular are complex and depend on an understanding of real fluid behaviour, often as only determined by experiment. This is the subject of Topic 5, Real Fluids. Pumps and turbines are explored later in this section as an application of the integral methods.