

Lecture 12: Conservation of Mass and Energy

Conservation of Mass: The Continuity Equation

An element of volume $d\forall$ has variable density but constant mass:

$$dm = \rho d\forall$$

A system is defined as a fixed mass (excluding losses eg to evaporation) hence its rate of change is zero:

$$\frac{\partial m_s}{\partial t} = \frac{D}{Dt} \int \rho d\forall = 0$$

(The notation $\frac{D}{Dt}$ is the Stokes' or material derivative, the rate of change in a Lagrangian system)

This can be related to the control volume via Reynolds' Transport Theorem (remembering once again that this is strictly correct only on average over the control volume):

$$\frac{\partial m_s}{\partial t} = \frac{\partial(\rho\beta)}{\partial t} + \nabla \cdot (\rho\beta \underline{\mathbf{V}}) = 0$$

But since β is mass, $\beta = \text{mass/unit mass} = \text{unity (1)}$ and we are left with:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{\mathbf{V}}) = 0 \quad \text{Mass Continuity Equation}$$

The equation in this form has the units of mass per time, averaged over the control volume. The operator $\nabla \cdot (\rho \underline{\mathbf{V}})$ is the divergence of a momentum field, and is equal/opposite to the dynamic rate of density change. If the flow is steady or incompressible, $\frac{\partial \rho}{\partial t} = 0$ and hence $\nabla \cdot (\rho \underline{\mathbf{V}}) = 0$.

Since $\nabla \cdot (\rho \underline{\mathbf{V}})$ is a surface flux (as per RTT derivation), we can use an equivalent surface integral:

$$\nabla \cdot (\rho \underline{\mathbf{V}}) = \Delta \rho (\underline{\mathbf{V}} \cdot \underline{\mathbf{A}}) = (\rho VA)_2 - (\rho VA)_1$$

Thus if $\nabla \cdot (\rho \underline{\mathbf{V}}) = 0$, then $(\rho VA)_2 = (\rho VA)_1 = \text{constant}$

or $\rho VA = \text{constant} = \text{mass flowrate } \dot{m} \text{ kg/s} \quad \text{Steady Continuity Equation}$

If the flow is incompressible, ρ is also constant and so

$$VA = \text{constant} = \text{volume flowrate } Q \text{ m}^3/\text{s} \quad \text{Incompressible Continuity Eqn}$$

With these equations it is possible to relate geometric changes (ΔA) to kinematic (ΔV) and dynamic ($\Delta \rho$) changes.

Example: A pipe 150 mm in diameter contracts down to 100 mm diameter. If an incompressible liquid flows at 1.5 m/s in the upstream section, what is its velocity in the downstream?

$$\text{The volume flowrate is } Q = VA = 1.5\pi(0.15)^2/4 = 0.03 \text{ m}^3/\text{s}$$

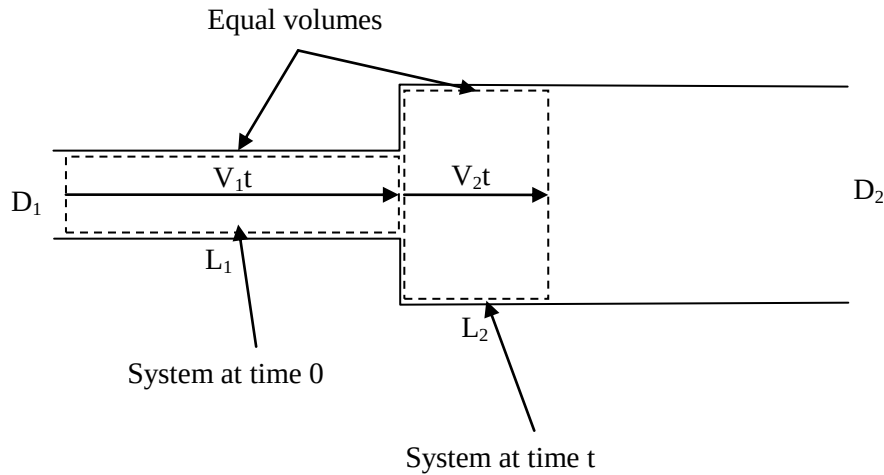
This is a constant for an incompressible fluid. In the downstream section $V_2\pi(0.1)^2/4 = 0.03$ or $V_2 = 3.375 \text{ m/s}$. In general for incompressible flows $V_2 = V_1 A_1/A_2$

Continuity: The easy derivation

The incompressible continuity equation can be derived by thinking about the geometry of a flow as it moves from a pipe of smaller to larger diameter. An incompressible fixed volume $\forall$ occupies two different lengths at successive moments in the two diameters:

$$L_1 = \forall / A_1 \quad (A_1 = \pi D_1^2/4)$$

$$L_2 = \forall / A_2 \quad (A_2 = \pi D_2^2/4)$$



The length change involves a step change in velocity as the plug moves through the junction. The velocity change can be derived by considering the time it takes the plug to flow in the two sections. The trailing end of the plug travels L_1 in the same time the leading end travels L_2 . Since $L = Vt$,

$$L_1 = V_1t, \quad L_2 = V_2t,$$

Divide L_1/L_2 to eliminate t and $\forall$:

$$L_1/L_2 = V_1/V_2 = A_2/A_1$$

Thus $V_1A_1 = V_2A_2$ or $VA = \text{constant}$.

This geometric derivation can be usefully compared to the mathematical one already given. In particular it provides an insight into the zero divergence, $\nabla \cdot (\rho \underline{V}) = 0$: The expansion from D_1 to D_2 involves an increase in the dimensions of the flow in that direction: $\nabla \cdot (\rho \underline{V}) = 0$ states that such an increase must be balanced by a corresponding (equal) *decrease* in the other directions.

Thus as the flow area increases (or decreases) the length of a simple incompressible flow system conversely decreases (or increases), as does the corresponding velocity.

This integral method compares one section with another and uses values at one section to derive values at the other. This is the general rule for all integral methods. The rule derives from the fact that the divergence is based on a surface integral and as such contains a comparison of the inflow (upstream) surface with the outflow (downstream) surface.

The geometric derivation does not easily adapt to compressible flow. The differential form of the equation is an essential (but incomplete) description of dynamic compression:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{V}) = 0$$

With the geometric role of the divergence understood, the rate of compression specifies additional geometric changes that can occur.

Conservation of Energy: The Euler and Bernoulli Equations

Compressible flows and many other technical problems are not soluble by the continuity equation alone. Additional information in the form of an energy transformation law is required. The full derivation is provided here, but you may want to start with the “easy” derivation in the next section.

We start with the *First Law of Thermodynamics*: $dU = dQ - dW$ (refer 1st year Physics courses)

Where d represents a change in a variable

dU is the change in total internal energy of a system – kinetic, potential, thermal and chemical

dQ is heat flow into the system

dW is mechanical work done by the system on its surroundings

The total internal energy of the system is transported as per RTT:

$$\frac{\partial U_s}{\partial t} = \frac{\partial Q}{\partial t} - \frac{\partial W}{\partial t} = \frac{D}{Dt} \oint \rho e dV$$

Where e is total internal energy per unit mass, including kinetic, potential and thermal energy:

$$e = \frac{V^2}{2} + gz + u$$

u is internal energy per unit mass. Neglecting chemical changes (not usually included in fluid calculations) u is proportional to temperature:

$$u \sim C_v T$$

where C_v is the fluid's heat capacity at constant volume and T is temperature in Kelvin.

Using Reynolds' Transport Theorem, recalling that it is only correct when averaged over the volume:

$$\frac{\partial U_s}{\partial t} = \frac{\partial Q}{\partial t} - \frac{\partial W}{\partial t} = \frac{\partial(\rho e)}{\partial t} + \nabla \cdot (\rho e \underline{\mathbf{V}}) \quad (1)$$

The heat flow term $\frac{\partial Q}{\partial t}$ is the rate of heat flow across the CV surface due to temperature differences.

The work rate term $\frac{\partial W}{\partial t}$ is the power of work done by the system on its surroundings.

Power can always be equated to the scalar product of the force and velocity vectors $\underline{\mathbf{F}}$ and $\underline{\mathbf{V}}$: fluid power includes power inputs and outputs such as pumps and turbines, generally referred to as *shaft work*, W_s :

$$\frac{\partial W}{\partial t} = -\underline{\mathbf{F}} \cdot \underline{\mathbf{V}} + \frac{\partial W_s}{\partial t}$$

ie the work rate is the resultant of forces acting on the control volume at the flow velocity, plus shaft work inputs.

The force resultant $\underline{F}$ arises from variable stresses $\underline{\tau}$ acting across the control volume:

$$\frac{\partial W}{\partial t} = -\oint \underline{\tau} \cdot \underline{V} \, d\mathbf{A} + \frac{\partial W_s}{\partial t}$$

$\underline{\tau}$ is a stress vector, a function of normal pressure and shear stress vectors: $\underline{\tau} = -P\hat{\mathbf{n}} + \underline{\tau}_s$

Where P (pressure) is positive in compression, τ_s is shear force and $\hat{\mathbf{n}}$ is a unit normal vector.

Thus
$$\frac{\partial W}{\partial t} = \oint P\hat{\mathbf{n}} \cdot \underline{V} \, d\mathbf{A} - \oint \underline{\tau}_s \cdot \underline{V} \, d\mathbf{A} + \frac{\partial W_s}{\partial t}$$

And from (1) above:

$$\begin{aligned} \frac{\partial U_s}{\partial t} &= \frac{\partial Q}{\partial t} - \frac{\partial W}{\partial t} = \frac{\partial Q}{\partial t} - \frac{\partial W_s}{\partial t} - \oint P\hat{\mathbf{n}} \cdot \underline{V} \, d\mathbf{A} + \oint \underline{\tau}_s \cdot \underline{V} \, d\mathbf{A} \\ &= \frac{\partial Q}{\partial t} - \frac{\partial W_s}{\partial t} - \nabla \cdot (P\underline{V}) + \oint \underline{\tau}_s \cdot \underline{V} \, d\mathbf{A} \\ &= \frac{\partial(\rho e)}{\partial t} + \nabla \cdot (\rho e \underline{V}) \quad \text{(using RTT for energy per unit mass, e)} \end{aligned}$$

The divergence terms for pressure work and energy flux can be added together: rearranging,

$$\frac{\partial Q}{\partial t} - \frac{\partial W_s}{\partial t} + \oint \underline{\tau}_s \cdot \underline{V} \, d\mathbf{A} = \frac{\partial(\rho e)}{\partial t} + \nabla \cdot \left(\rho \left(e + \frac{P}{\rho} \right) \underline{V} \right)$$

Where
$$e = \frac{V^2}{2} + gz + u$$

The loss terms can now be identified and collected together: all terms representing unusable forms of energy:

Shear stress losses (laminar and turbulent): $\oint \underline{\tau}_s \cdot \underline{V} \, d\mathbf{A}$

Heat loss $\frac{\partial Q}{\partial t}$

Internal energy changes and flux $\frac{\partial(\rho u)}{\partial t} + \nabla \cdot (\rho u \underline{V})$

Thus
$$-\frac{\partial W_s}{\partial t} = \frac{\partial}{\partial t} \left(\rho \left(\frac{V^2}{2} + gz \right) \right) + \nabla \cdot \left(\rho \left(\frac{V^2}{2} + gz + \frac{P}{\rho} \right) \underline{V} \right) + \text{losses}$$

Or
$$\frac{\partial}{\partial t} \left(\rho \left(\frac{V^2}{2} + gz \right) \right) + \nabla \cdot \left(\rho \left(\frac{V^2}{2} + gz + \frac{P}{\rho} \right) \underline{V} \right) + \frac{\partial W_s}{\partial t} + \text{losses} = 0.$$

Euler (unsteady compressible energy flux) equation

For steady flow the $\frac{\partial}{\partial t}$ derivative is zero. For uniform flow velocity the flux form of divergence gives

$$\Delta \left(\rho \left(\frac{V^2}{2} + gz + \frac{P}{\rho} \right) \mathbf{V} \cdot \mathbf{A} \right) + \frac{\partial W_s}{\partial t} + \text{losses} = 0$$

$$\rho_2 V_2 A_2 \left(\frac{V_2^2}{2} + gz_2 + \frac{P_2}{\rho_2} \right) - \rho_1 V_1 A_1 \left(\frac{V_1^2}{2} + gz_1 + \frac{P_1}{\rho_1} \right) + \frac{\partial W_s}{\partial t} + \text{losses} = 0$$

For incompressible flow we can divide by a constant $\rho V A g = \dot{m} g$ to obtain:

$$\frac{V_2^2}{2g} + z_2 + \frac{P_2}{g\rho} + \frac{\partial W_s}{\partial t} \frac{1}{\dot{m}g} + \frac{\text{losses}}{\dot{m}g} = \frac{V_1^2}{2g} + z_1 + \frac{P_1}{g\rho}$$

Shaft work per unit weight = shaft head, which we divide into energy inputs from pumps, H_p , and energy outputs to turbines, H_t . Losses per unit weight are defined as *headloss*. So we arrive at:

$$\frac{V_1^2}{2g} + z_1 + \frac{P_1}{g\rho} + H_p = \frac{V_2^2}{2g} + z_2 + \frac{P_2}{g\rho} + H_t + \text{headloss}$$

Bernoulli Equation (incompressible, steady, uniform velocity profile)
(Daniel Bernoulli, 1738)

Bernoulli's Equation: The easy derivation

Bernoulli's equation can also be derived from the integral forms of kinetic and potential energy, familiar from first-year physics as

$$KE = \frac{1}{2} m V^2 \quad \text{and} \quad PE = mgz \quad \text{Units Joules or N.m}$$

Both forms of energy are relative to a particular reference frame: a *datum* or reference height $z = 0$ and a system of coordinates with relative velocity $V = 0$. Within the local reference frame PE and KE are interchangeable: kinetic converts to potential and vice versa as an element of mass accelerates and moves up or down in the gravitational field.

An "element of mass" within a flow is a partly relative idea, since we can consider as much of the flow as we wish and the total volume of the flow is not fixed. It is usual to consider the *energy per unit volume* of the flow, a variable quantity related to the density and hence definable at any point:

$$\frac{KE}{\text{Volume}} = \rho \frac{V^2}{2} \quad \text{and} \quad \frac{PE}{\text{Volume}} = \rho g z \quad \text{Units Joules/m}^3 \text{ or N/m}^2 \text{ (Pa)}$$

The formulation of potential energy per unit volume is recognisable as hydrostatic pressure. At the datum where $z = 0$ the hydrostatic pressure P is at a maximum while the conventional potential energy is zero. At the surface where $z = H$ the converse occurs: hydrostatic pressure is zero and potential energy is at a maximum. Hydrostatic pressure and potential energy are thus interchangeable.

As we saw in Section 1, fluid pressure is a form of stored elastic energy, physically contained in the fluid as a result of compressive volume changes. As an element of fluid falls in depth the overlying weight force does work on the fluid over the compressive length, storing elastic energy. As the element rises, the same stored energy is conservatively released, in the same proportion as the potential energy increases.

Fluid energy thus translates between three forms: kinetic, potential and pressure energy:

$$\frac{\text{Energy}}{\text{Volume}} = \rho \frac{V^2}{2} + \rho g z + P = \text{conserved (constant)}$$

The mathematical derivation of Bernoulli's equation confirms that pressure derives from the system, while kinetic and potential energy derive from the control volume.

The full derivation also shows that Bernoulli's equation is in fact *not* constant but is subject to non-conservative losses, which can be formulated in thermodynamic terms, and may also be effected by inputs and outputs of energy via pumps and turbines:

$$\rho_1 \frac{V_1^2}{2} + \rho_1 g z_1 + P_1 + \text{inputs} - \text{outputs} - \text{losses} = \rho_2 \frac{V_2^2}{2} + \rho_2 g z_2 + P_2$$

A more complete formulation of losses depends on an understanding of real fluid behaviour (Section 5). Pump and turbine actions require more detail from fluid momentum theory (Section 4.4).

For incompressible fluids Bernoulli's equation is often stated in units of pressure / unit weight, which has the same units as height: to see this, just divide both sides by a constant ρg :

$$\frac{V_1^2}{2g} + z_1 + \frac{P_1}{\rho g} + \text{PumpHead} - \text{TurbineHead} - \text{HeadLoss} = \frac{V_2^2}{2g} + z_2 + \frac{P_2}{\rho g} \quad \text{units m}$$

Non-uniform flow profiles

A uniform flow profile provides a simple solution of the divergence. Non-uniform profiles fit the same distribution so long as a *kinetic energy correction factor* is included:

$$\alpha = \frac{\int V^3 dA}{\bar{V}^3} \quad \text{where } \bar{V} \text{ is the mean flow velocity.}$$

α has no units and is applied as a *coefficient* or multiplier:

$$\frac{\alpha_1 V_1^2}{2g} + z_1 + \frac{P_1}{\rho g} + H_p = \frac{\alpha_2 V_2^2}{2g} + z_2 + \frac{P_2}{\rho g} + H_t + k \frac{V^2}{2g}$$

For most internal turbulent flows the velocity distribution is nearly uniform and energy correction factor $\alpha = 1.05$. For laminar (parabolic) flow, $\alpha = 2.0$.

Fluid Power

The energy carried by a fluid is continuous and can be regarded as a form of power, ie a steady rate of energy flow. Furthermore any gradual or sudden changes in pressure or velocity terms of Bernoulli's equation equate to the power of that exchange as a continuous rate of input or extraction of energy.

Power can be expressed and formulated in a number of different ways. The basic form is of Force x Velocity, where the force term is the net pressure force or reaction force due to an exchange and the velocity is the speed at which this occurs:

$$\text{Power} = \text{Force} \times \text{Velocity} \quad \text{Basic form of fluid power (Units J/s or Watts)}$$

This can be converted to a form more useful in practice for a known pressure change ΔP :

$$\text{Force} = A \Delta P = A \rho g \Delta H \quad \text{and} \quad \text{Velocity} = Q/A$$

$$\text{Thus} \quad \text{Power} = \rho g Q \Delta H = Q \Delta P \quad \text{Head or Pressure Loss form of power (W)}$$

Bernoulli's Equation: A Relationship between Pressure and Velocity

The main value of Bernoulli's equation is that it provides a relationship between pressure and velocity, often the key knowns and unknowns in fluid problems. These two terms can be considered alone, since losses and elevation are often negligible over short ranges:

$$\rho \frac{V^2}{2} + P = \text{constant}$$

Pressure-velocity relationship, units Pa

Thus as velocity increases, pressure decreases, and vice versa. This effect is common to all forms of non-uniform fluid flow and produces a range of environmentally and technologically important effects.

Where a problem deals directly with unknown pressures it makes sense to leave the equation in the units of pressure as shown. Often however we need to compare flow energy levels with the elevations of infrastructure or ground levels, in which case a conversion to metres of head is useful. Dividing by ρg :

$$\frac{V^2}{2g} + \frac{P}{\rho g} = \text{constant}$$

Head-velocity relationship, units m

A check on the units is useful:

$$\frac{V^2}{2g} = \left[\frac{m}{s} \right]^2 \left[\frac{s^2}{m} \right] = \left[\frac{m}{1} \right] \quad \text{and} \quad \frac{P}{\rho g} = \left[\frac{N}{m^2} \right] \left[\frac{m^3}{kg} \right] \left[\frac{s^2}{m} \right] = \left[\frac{kg \cdot m}{m^2 \cdot s^2} \right] \left[\frac{m^3}{kg} \right] \left[\frac{s^2}{m} \right] = \left[\frac{m}{1} \right]$$

This applies only where losses and elevation changes are negligible. In general a significant elevation change will also be associated with significant losses, so the two go together. With losses it becomes less useful to identify a constant: the equation is more usually stated as a comparison between sections 1 and 2:

$$\frac{V_1^2}{2g} + \frac{P_1}{\rho g} + z_1 - \text{losses} = \frac{V_2^2}{2g} + \frac{P_2}{\rho g} + z_2$$

Elevation and loss factors obscure the relationship between pressure and velocity, but it still applies in general terms.

A video <http://www.youtube.com/watch?v=kXBxtaf2TTg&feature=related> further explains and illustrates Bernoulli's equation.



Thought Experiment: Emptying the Bath

If you ever run out of petrol on a windy stretch of highway you'll wish you'd practiced the art of *siphoning*, or making a liquid flow under its own suction head from one tank to another. You can practice this useful trick in the bath, or just think about it next time you're floating there half asleep. Can you empty your bath out of the window with no more energy than it takes to create a few feet of suction, applied only at the start of the process? How does this work? Try drawing a diagram of pressure and elevation. Be careful not to swallow the soapy stuff, or the petrol!

Reading

Potter and Wiggert 1991 (Reading 6) Section 4.3 and 4.4