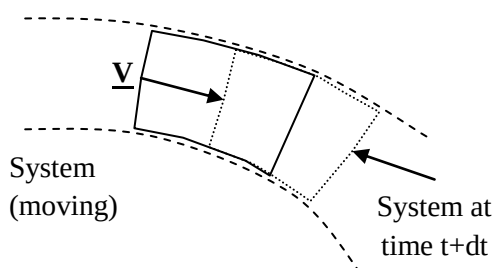


Lecture 11: Reynolds Transport Theorem

A concept was introduced in Section 3 that the conserved quantities of mass, energy and momentum are measurable properties of fluid *systems*, that is of particular fluid masses or volumes moving through a flow field. Systems retain mass and transform energy among a limited number of modes, or exchange momentum with boundaries and obstructions in the flow. These exchange processes can be accounted for by applying the conservation laws.

We can use these principles to work out reliable relationships between the key dynamic variables of flow: the geometry and rise or fall of the conduit, the velocity and pressure, and the forces and energy exchanges taking place between the flow and its surroundings. This leads to important practical applications including flow measurement, water distribution, drainage, pumps, aircraft and turbine machinery. It also provides solution conditions for a range of important environmental effects.

To apply the conservation laws we frequently need a conversion from a Lagrangian (moving) to an Eulerian (fixed) reference frame, that is from a system in a moving reference frame to a *control volume* in a stationary reference frame, relative to the boundaries of the flow. The control volume or fixed reference frame is usually more convenient as a framework for calculating unknowns from knowns, such as downstream pressures estimated from upstream measurements:

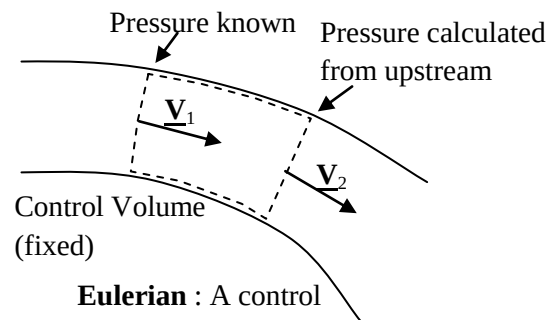


Lagrangian : A system has conserved properties that can be accounted for:

Mass = m

Momentum = $m\underline{V}$

Energy = $mV^2/2$ among other flow properties



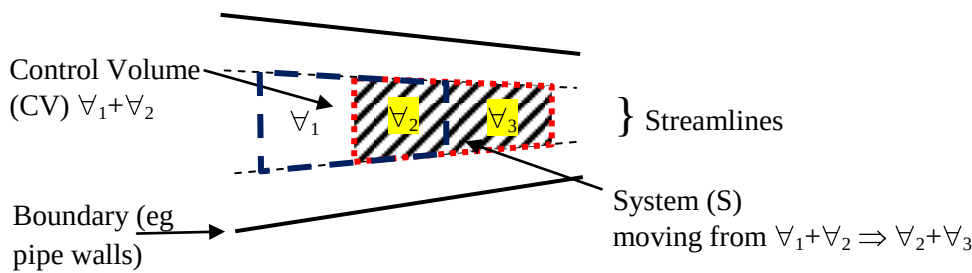
Eulerian : A control volume is a useful fixed reference frame for comparing and solving upstream or downstream values using integral methods.

The Eulerian description is obtained from the Lagrangian by a *transformation*, based in part on the Gauss Divergence Theorem (see Vector Calculus Refresher.pdf in the Tutorials), known as *Reynolds' Transport Theorem*:

Like the Divergence Theorem, Reynolds' Transport Theorem (RTT) is a geometric description of the relationship between inflow, outflow and storage within a volume. The power of RTT is that it compares two types of volumes, a system and a control volume, and derives a relationship between them that provides a conversion from moving (Lagrangian) to fixed (Eulerian) reference frames.

Reynolds' Transport Theorem: The easy derivation

A system moves through a control volume, at one moment completely occupying the CV. Using streamlines to outline the volumes and the movement of the system we can sketch a comparison between the system and control volume at the moment of overlap and at some time later:



Conservation and transportation of a system through a control volume.

At time t the system and CV overlap: $S(t) = CV(t) = \nabla_1 + \nabla_2$

At time $t + \Delta t$, the system has moved to $S(t + \Delta t) = \nabla_2 + \nabla_3$

The control volume remains the same: $CV(t + \Delta t) = \nabla_1 + \nabla_2$

In the time interval Δt the volume ∇_3 leaves the CV and is replaced by ∇_1

Any conserved quantity B (could be mass, momentum, energy) transported by the flow is a property of both the system and the control volume:

$B = \rho\beta\nabla$ where ∇ is volume, ρ is fluid density and β is the concentration of B per unit mass of fluid.

In the time interval Δt , the quantity B may change by an amount ΔB :

$$\Delta B = B' - B = B_{t+\Delta t} - B_t$$

where $'$ indicates the changed (new) value of a parameter (pronounced “prime” eg $B' = \text{“B-prime”}$). The changing quantity is similar for the CV and the system, with important differences:

$$\Delta B_{cv} = [(\rho'\beta')_1 + (\rho'\beta')_2] - [(\rho\beta)_1 + (\rho\beta)_2] \quad \text{Control Volume change}$$

B in ∇_1 and ∇_2 at time $t + \Delta t$ B in ∇_1 and ∇_2 at time t

$$\Delta B_s = [(\rho'\beta')_2 + (\rho'\beta')_3] - [(\rho\beta)_1 + (\rho\beta)_2] \quad \text{System change}$$

B in ∇_2 and ∇_3 at time $t + \Delta t$ B in ∇_1 and ∇_2 at time t

Note that the two changes are very similar, except for the terms $(\rho'\beta')_1$ and $(\rho'\beta')_3$ at $t + \Delta t$: We can compare the two changes by a straightforward subtraction of one equation from the other:

$$\Delta B_s = \Delta B_{cv} + [(\rho'\beta')_3 - (\rho'\beta')_1]$$

System Control Volume

Amount that has left CV in Δt Amount that has entered CV in Δt

Basic relationship between system and CV

This is an example of a storage-discharge relationship: for a given flowrate there is a corresponding volume in transit. For example a reservoir overflows at a rate related to the volume of water over the level of the spillway.

The amounts that enter and leave the CV equate to the total flux (inflow + outflow) across the CV surfaces:

$$((\rho'\beta')_3 - (\rho'\beta')_1)\Delta t = \Delta t \oint \rho\beta \underline{\mathbf{V}} \cdot \underline{\hat{\mathbf{n}}} dA$$

where $\rho\beta \underline{\mathbf{V}} \cdot \underline{\hat{\mathbf{n}}}$ is a vector flux of B carried by $\underline{\mathbf{V}}$ over the surface. The surface integral depends on the specific shape of inflow and outflow surfaces, but can be simplified by application of Gauss' theorem (see vector calculus refresher):

$$\oint \underline{\mathbf{V}} \cdot \underline{\hat{\mathbf{n}}} dA = \iiint \nabla \cdot \underline{\mathbf{V}} d\forall$$

Thus

$$\Delta B_s = \Delta B_{cv} + \Delta t \iiint \nabla \cdot (\rho\beta \underline{\mathbf{V}}) d\forall$$

Which relates the changing system and control volume to the divergence of the flow. Note that the amounts ΔB_s and ΔB_{cv} are also volume integrals. Using partial differentials:

$$\begin{aligned} \frac{\partial B_s}{\partial t} &= \frac{\partial B_{cv}}{\partial t} + \iiint \nabla \cdot (\rho\beta \underline{\mathbf{V}}) d\forall \\ &= \frac{\partial}{\partial t} \iiint \rho\beta d\forall + \iiint \nabla \cdot (\rho\beta \underline{\mathbf{V}}) d\forall \end{aligned}$$

Eliminating the volume integrals for any point in the system or control volume,

$$\frac{\partial B_s}{\partial t} = \frac{\partial(\rho\beta)}{\partial t} + \nabla \cdot (\rho\beta \underline{\mathbf{V}}) \quad \text{Reynolds' Transport Theorem}$$

This theorem relates the conservative physical changes that take place within a system to the geometric and kinematic changes that occur in the control volume. We can use physical principles such as Newton's second law or the conservation of mass to define the system's changes (or its constants), and solve for the kinematics of the flow as they change from section to section. All terms have the units of the quantity B per time in motion (transport) past a point.

Strictly speaking this *differential* form of the theorem only applies when averaged over a unit volume of fluid: having eliminated the volume integrals we are left with an equation that should apply at any given point in the flow, but is dimensionally correct only when averaged over the volume.

The *integral methods* that are developed in the rest of this section get past this problem by considering the changes that occur between the two ends of a control volume. If conditions at one location in the fluid are known, corresponding conditions at other points upstream or downstream of that point can be calculated. This provides solution methods that match not only the types of problems we must solve in practice but also match the limited data that is usually available.

Importantly, the theorem splits the total change between temporal (d/dt) and spatial (∇) components. The temporal change is sometimes referred to as *local*, in that any time change can be said to occur at specific locations. The spatial change is referred to as *convective*, in that motion from place to place is a transport mechanism for quantities such as mass, momentum and energy.

The convective operator $\nabla \cdot (\rho\beta \underline{\mathbf{V}})$ is a flux term derived from the surface integral. An alternative expression that compares upstream and downstream sections (subscripts 1 and 2 in that order) is:

$$\nabla \cdot (\rho\beta \underline{\mathbf{V}}) = \Delta(\rho\beta \underline{\mathbf{V}} \cdot \underline{\mathbf{A}}) = (\rho\beta V A)_2 - (\rho\beta V A)_1$$

This leads to integral methods by which two locations (subscripts 1 and 2) can be compared. You can relax now! The strange vector operator ∇ (pronounced DEL) is only shown for completeness, it is really just an expression of the differences Δ (pronounced DELTA) between two points.

β can be any quantity per unit mass transported by any mechanism between two sections 1 and 2:

- Momentum
- Thermal Energy

- Solutes, contaminants, sediment
- Kinetic Energy

The key conserved quantities lead to three widely used integral methods:

1. Mass conservation: Continuity equation
2. Energy conservation: Bernoulli equation
3. Momentum conservation: Force Reaction laws

Reading

Potter and Wiggert 1991 (Reading 6) Section 4.1 and 4.2