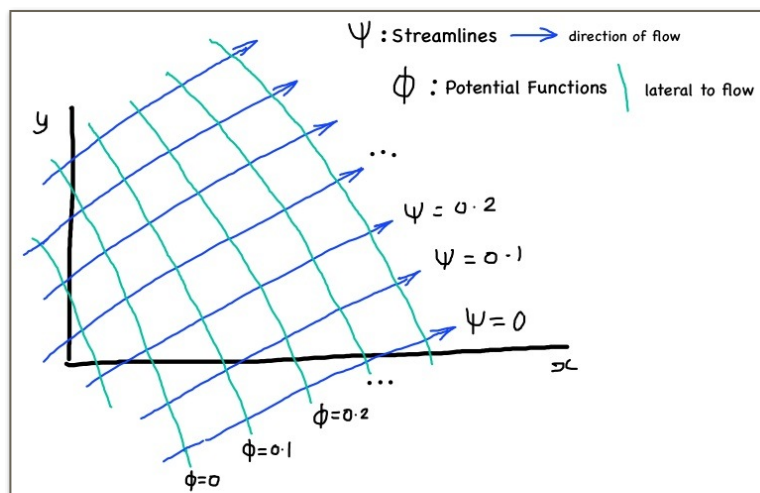


Potential Flow

Stream functions and flow nets

2-dimensional ideal fluid flow (incompressible, irrotational, inviscid) can be represented as a flow net combining orthogonal *streamlines* and *potential functions*.

In two dimensions we can picture an ideal flow with a velocity distribution. In plan view, looking down on the x-y plane:



Streamlines indicate the direction of flow, while potential lines can be interpreted as lines of equal energy or head. As will be seen, there is a natural physical interpretation of ψ , but ϕ is more abstract and can be taken as a method for solution of ψ .

The flow net is *orthogonal*: ϕ and ψ functions always meet at right angles.

A streamline is defined as tangential to the flow velocity at every point, so flow cannot cross a streamline. This is recognisable as a law of continuity, in that the flowrate is constant between any pair of streamlines.

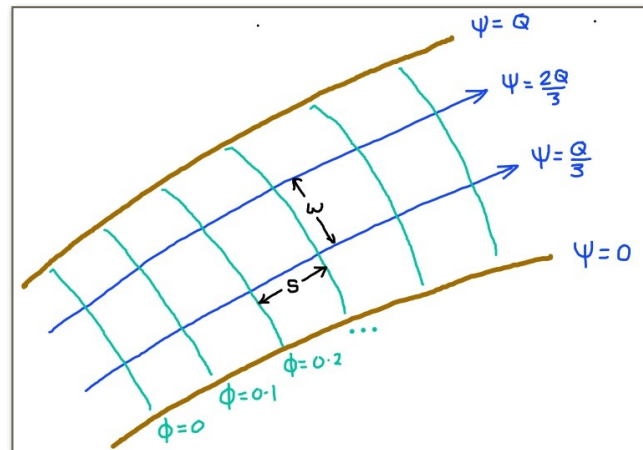
The stream function ψ is defined as the incremental flowrate as it increases, from zero at a reference streamline such as a wall or boundary, to the flowrate q at any particular streamline:

$$\psi = q \quad q \text{ the flowrate between boundary and the streamline of that value.}$$

Increments of ψ represent the fraction of flow rate between any pair of stream lines. For this reason ψ always increases in a lateral direction, usually taken as to the left of forward motion.

Similarly the value of the potential function ϕ increases in the forward direction of flow. ϕ is not easily defined, other than that it is orthogonal to ψ and this leads to useful solutions.

For example, a flow in a 1 m deep diverging rectangular open channel divided into three distinct flow bands with a total flow of Q might look like this:



The flow takes this orthogonal form because it is an ideal fluid, ie with no viscous friction, so potential lines are not sheared by the boundary.

The orthogonal grid of a flow net can be solved mathematically or drawn by hand. The manual technique is to draw, erase and redraw a flow net until each grid cell is approximately square. A hand-drawn flow net can reproduce a mathematically correct flow net.

Defining the length and width of a square element as s and w respectively, and using continuity:

$Q = VA = \text{Constant increment of flow between streamlines.}$

But $A = w$ per unit depth into page (2D flow).

Thus $q = V \cdot w = \Delta\psi$ or $V = \Delta\psi/w$.

In the limit as w goes to zero,

$$V = \frac{\partial\psi}{\partial w}$$

ie the flow velocity is equal to the rate of change of ψ in the lateral, or width direction. In 2 dimensions:

$$V_x = \frac{\partial\psi}{\partial y} ; V_y = - \frac{\partial\psi}{\partial x}$$

The negative sign indicates that V_y tends to decrease on any pathway parallel to the x -axis, assuming positive flow to the right and up, ie it is a directional sign convention.

Orthogonality arises when the potential function ϕ increases at the same rate as the stream function ψ . So as ψ diverges or converges, so does ϕ . As a result, the continuity law relating velocity to discharge also applies to ϕ , but in the forward (length, or s) direction:

$$V = \frac{\partial\psi}{\partial w} = \frac{\partial\phi}{\partial s}$$

And again in 2 dimensions,

$$V_x = \frac{\partial \phi}{\partial x} ; V_y = \frac{\partial \phi}{\partial y}$$

That is, the velocity distribution of an ideal flow is specified by the stream and potential functions.

Basic relationships between ψ and ϕ can be derived by considering the derivative equations of incompressible and irrotational flow. For incompressibility we have:

$$\nabla \cdot (\rho V) = 0 \quad (\text{see the tutorial "Vector Calculus for Fluid Mechanics"})$$

or

$$\nabla \cdot V = 0$$

This gives an incompressible flow with zero divergence, ie for any increase in velocity in a given direction there is a compensating decrease in other directions.

That is;

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0$$

or

$$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

Since cross-derivatives are almost always equal, *most stream functions are continuous* in the ideal fluid sense. **Almost any well-behaved vector function could be a stream function.**

We also have

$$\nabla \times V = 0$$

This gives zero vorticity, or an irrotational flow.

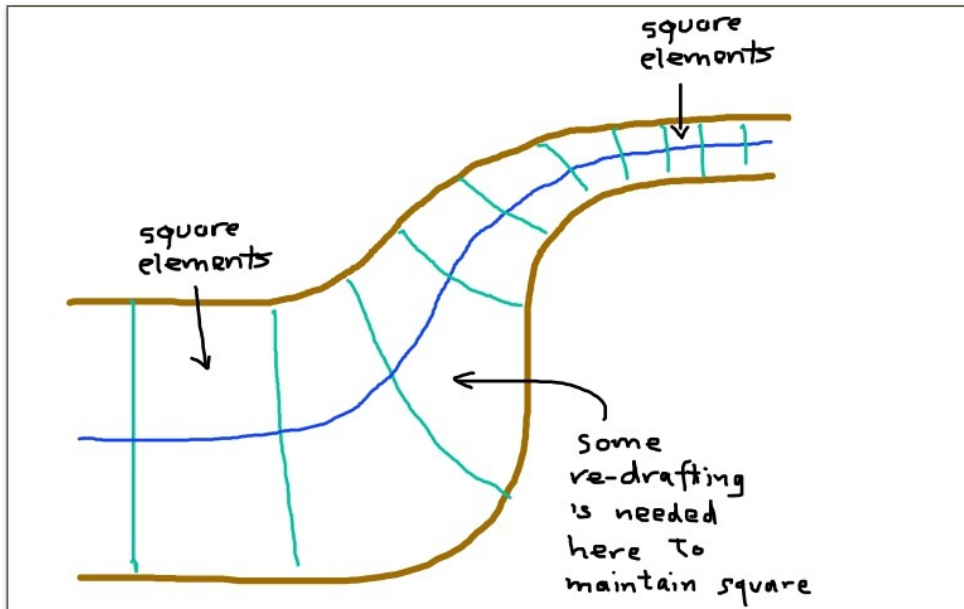
That is;

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

This is a more unique condition which tends to set limits on suitable stream functions. It can be abbreviated as:

$$\nabla^2 \psi = 0$$

The Laplace Equation



Usefully, the potential function ϕ is also determined by the Laplace equation. Since divergence is zero, and with the same continuity and vorticity constraints as before:

$$V_x = \frac{\partial \phi}{\partial x} ; V_y = \frac{\partial \phi}{\partial y}$$

and

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0$$

we find

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Laplace Equation

similarly

$$\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x} = 0$$

So again most potential functions are continuous in the ideal fluid sense. The Laplace equation is the main determining condition.

'Del-squared' Equations of Stream and Potential Flow

$$\nabla^2 \psi = 0$$

$$\nabla^2 \phi = 0$$

The interpretation of these equations is that ideal flow has a balanced curvature of ψ and ϕ . You can picture them as mathematical surfaces like landscapes plotted above the x-y plane. Potential and streamlines are *contours* of these surfaces, ie lines of constant flowrate or potential at constant height above the plane.

ψ and ϕ increase monotonically (ie they do not turn around and begin to decrease again) in the downstream and lateral directions, and they have matching variable rates of increase. These properties can be used in either of two modes:

- A. Analytical, by solving the equations with boundary conditions to find the velocity distribution;
or
- B. Graphical, by sketching solutions that have the property of mutual orthogonality.

A few points illustrate how stream and potential functions work:

1. ψ is constant along a streamline:

$$\partial\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy = 0$$

thus

$$\frac{\partial\psi}{\partial x}dx = -\frac{\partial\psi}{\partial y}dy$$

or

$$\frac{\partial y}{\partial x} = -\frac{\frac{\partial\psi}{\partial x}}{\frac{\partial\psi}{\partial y}} = \frac{V_y}{V_x}$$

This indicates that velocity is tangential to a streamline - the slope of each streamline is equal to the ratio of velocities (this is the definition of a streamline).

2. ϕ is constant along a potential function:

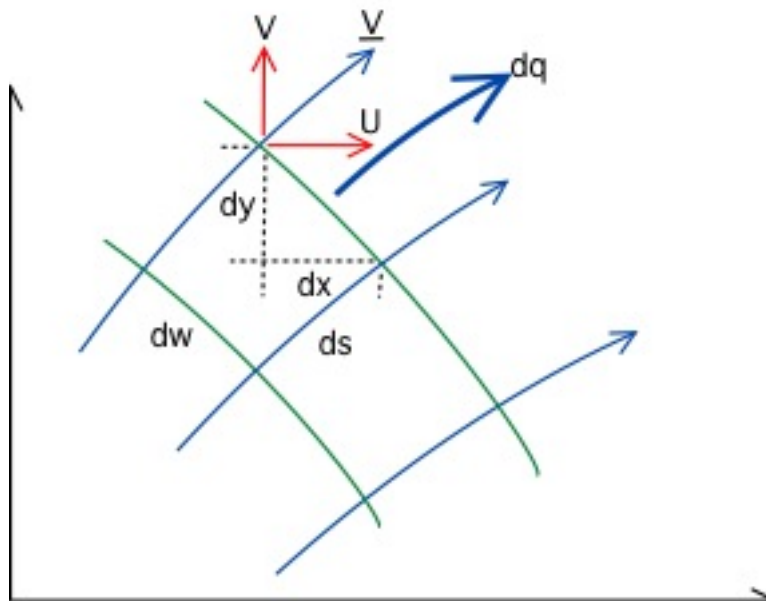
$$\partial\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy = 0$$

$$\frac{\partial\phi}{\partial x}dx = -\frac{\partial\phi}{\partial y}dy$$

$$\frac{\partial y}{\partial x} = -\frac{\frac{\partial\phi}{\partial x}}{\frac{\partial\phi}{\partial y}} = -\frac{V_x}{V_y}$$

which indicates that ϕ is orthogonal (with negative reciprocal slope) to the streamlines.

3. Flow rate q and stream line ψ have equal increments between streamlines:



dq = increment of flow rate between stream lines.

$$dq = \underline{V}.dw = V(-dx) + U(dy)$$

Where V is the vertical velocity component and U is the horizontal. The x increment is negative due to the same directional sign convention as before.

but

$$V_x = \frac{\partial \psi}{\partial y} ; V_y = -\frac{\partial \psi}{\partial x}$$

so

$$\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = dq$$

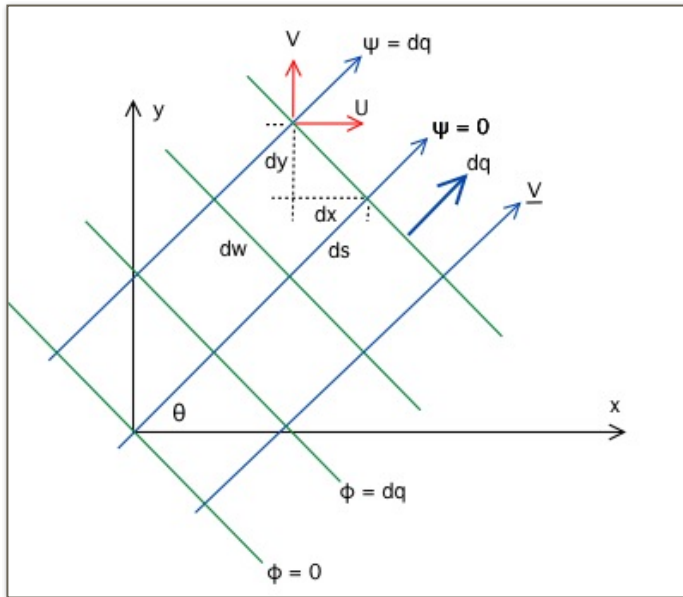
or more simply

$$d\psi = dq$$

This is the definition of the value of the streamline function ψ , or its derivative.

Potential flow and stream functions can be used to analyse a range of useful 2D fluid flows. All potentiometric solutions are approximate as they do not include the effects of viscosity, mainly turbulence. Because of this approximation, *there is no correct answer*; candidate solutions can be based on utility and tractability rather than ideal physical accuracy.

Model 1: Uniform rectilinear flow



Taking $(\psi = 0, \phi = 0)$ to coincide with $(x, y) = (0, 0)$ (the choice of datum is often arbitrary) we have

$$\begin{aligned}\partial\psi &= \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy = -V_ydx + V_xdy \\ &= Udy - Vdx\end{aligned}$$

thus

$$\begin{aligned}\psi &= \int U dy - V dx \\ &= Uy - Vx\end{aligned}$$

This equation can be solved for y as a function of x by treating the value of ψ as a free parameter. For any constant value of ψ (ie along a streamline) the streamline plots as:

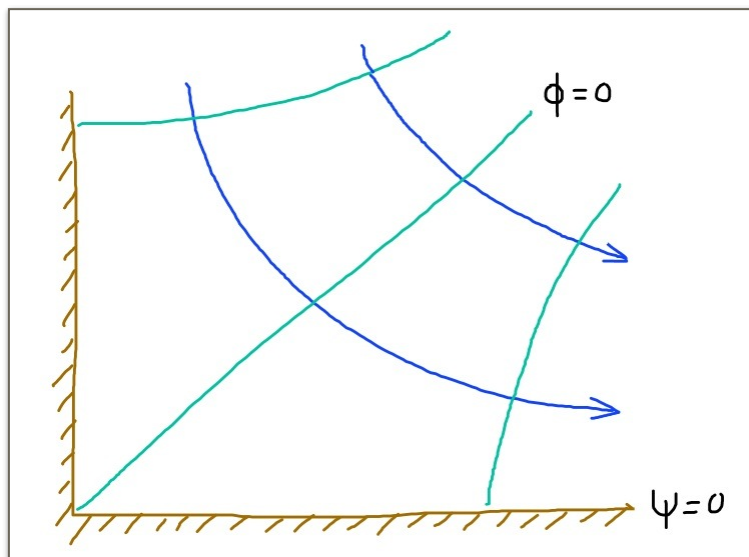
$$\begin{aligned}y &= \frac{\psi + Vx}{U} \\ &= \frac{\psi}{U} + x \tan\theta\end{aligned}$$

This equation can be plotted as a family of straight lines representing streamlines for arbitrary values of $\psi = 0, 0.1$ etc.

In more complex situations the potential function ϕ can also be evaluated and treated as a free parameter in y as a function of x . This often gives useful additional insights.

Model 2: An open corner

Use boundary conditions to derive a simple model of potential flow in a 90° corner. Check the model for incompressibility and irrotationality.



An approximate flow net can be sketched to develop the idea.

The value of $\phi = 0$ is again arbitrary, but the natural value of $\psi = 0$ lies along the boundary, since there is zero flowrate along that line. For simplicity the corner is taken as the $x - y$ axes ie $x = 0, y = 0$.

External conditions in the open flow are not specified, so this model will not be realistic for large values of x or y .

In the closed corner, ie along the boundary, the velocities are determined by the fact that flow does not cross the boundary. These are the boundary conditions:

$$U = 0 \text{ @ } x = 0 \text{ and } V = 0 \text{ @ } y = 0.$$

With simplicity in mind, and ignoring the influence of large values of x and y , the model may be as simple as:

$$U = k.x \quad V = -k.y$$

or

$$\underline{V} = (kx, -ky)$$

The vertical velocity V is always negative in this descending flow net. This model satisfies the boundary conditions and has the correct sense or sign in the first quadrant.

The stream function can be derived from this model of velocity:

$$V_x = \frac{\partial \psi}{\partial y} = kx \quad ; \quad V_y = -\frac{\partial \psi}{\partial x} = -ky$$

both cases have the same solution:

$$\frac{\partial \psi}{\partial y} = kx \quad \frac{\partial \psi}{\partial x} = -ky$$

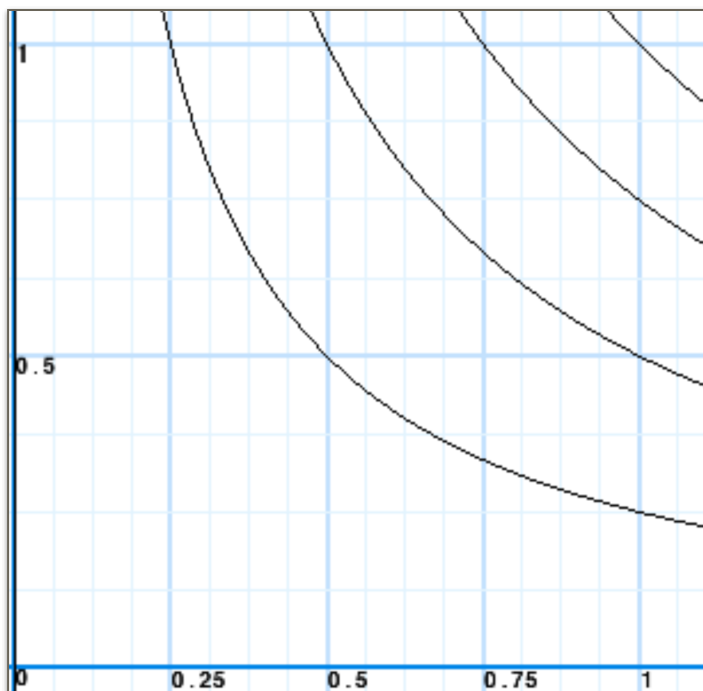
or

$$\psi = kxy$$

thus

$$y = \psi/kx$$

The streamlines plot in Cartesian (x, y) coordinates as a family of reciprocal curves inversely proportional to x. For $k = 1$ and $d\psi = 0.25$ the first four streamlines ($\psi = 0.25$ to 1.0) plot in the corner ($\psi = 0$ along the axes):



The potential functions are plotted in a similar way:

$$V_x = \frac{\partial \phi}{\partial x} ; V_y = \frac{\partial \phi}{\partial y}$$

so

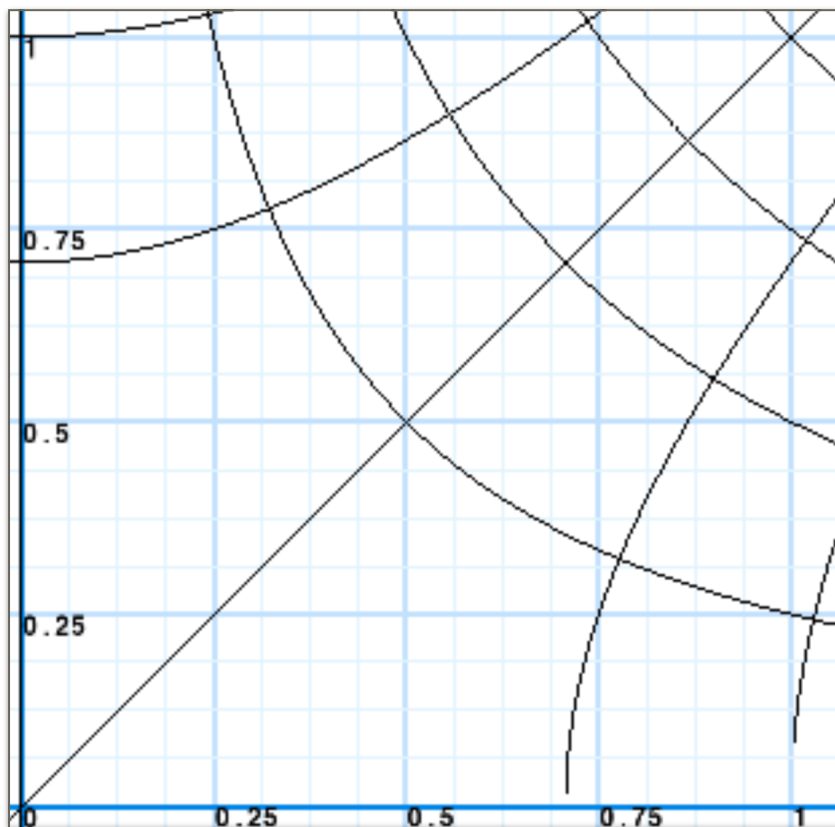
$$\partial \phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy$$

$$= kx dx - ky dy$$

$$\phi = \frac{kx^2}{2} - \frac{ky^2}{2}$$

$$y = \sqrt{\frac{2}{k} \left(\frac{kx^2}{2} - \phi \right)} = \sqrt{(x^2 - 2\phi/k)}$$

this plots as the diagonal $y = x$ for $\phi = 0$, or a family of power curves:



Incompressibility and irrotationality are tested by:

$$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

and

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

which is true given that

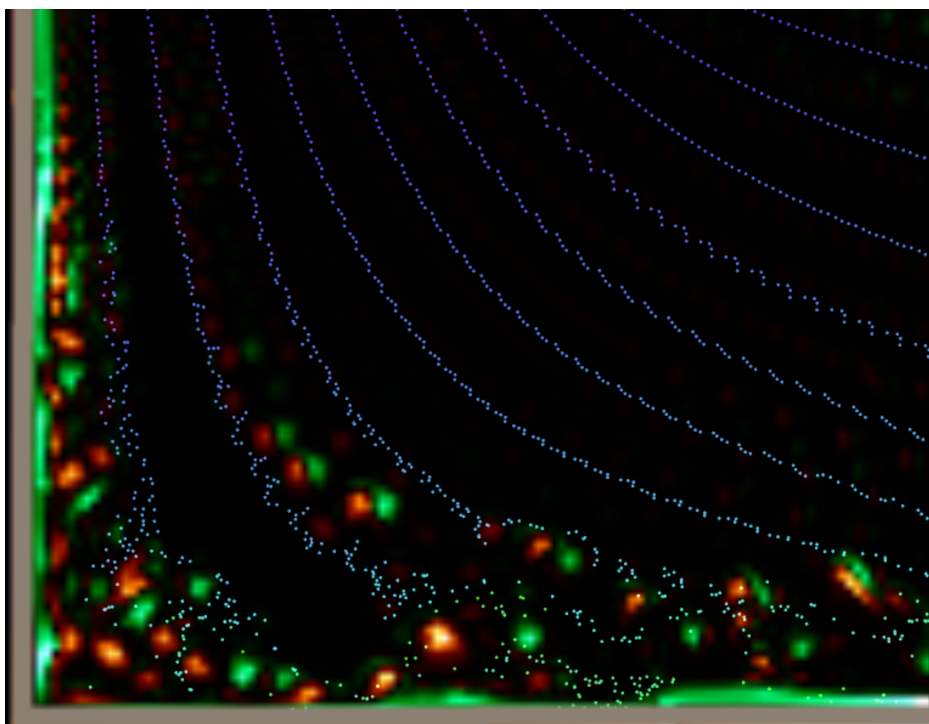
$$\psi = kxy$$

$$\frac{\partial \psi}{\partial x} = ky \quad ; \quad \frac{\partial \psi}{\partial y} = kx$$

$$\frac{\partial^2 \psi}{\partial x^2} = 0 \quad ; \quad \frac{\partial^2 \psi}{\partial y^2} = 0$$

$$\frac{\partial^2 \psi}{\partial x \partial y} = k \quad ; \quad \frac{\partial^2 \psi}{\partial y \partial x} = k$$

thus even this simple model is acceptable as ideal flow. However, a more accurate real flow solution with viscosity reveals the true complexity of the flow, showing rotational vortices (red/green for positive/negative curl) and bands of turbulent instability:



Linear Superposition and non-linear transformation

More complex flow patterns can be developed by adding together or *superposing* any number of simple solutions, and by applying mathematical tricks such as transformation (stretching and twisting) of the coordinate system.

For example a realistic sink vortex can be generated by superposing a radial sink (radial flow lines exiting at the centre) with an irrotational free vortex (flow in concentric circles).

Similarly a cylindrical obstruction in a flow can be constructed by combining a radial sink with a source (the opposite of a sink) in a uniform flow field. The shape of the obstruction can be modified to create a simple aerofoil (a Joukowski aerofoil) by applying a complex number transformation to the flow field.

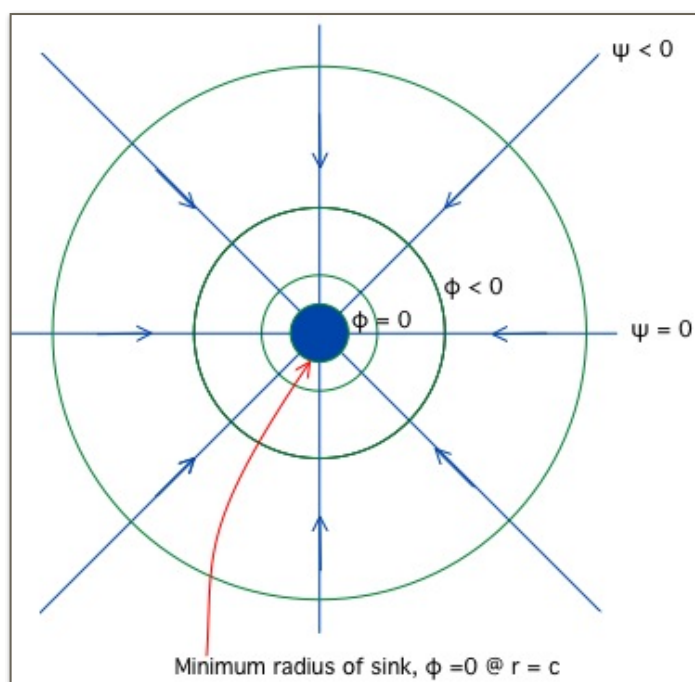
These are historically significant early solutions to flows of important theoretical and practical significance. The solutions are determinate, ie they produce soluble equations rather than differentials. This allowed flows to be analysed without extensive numerical analysis in the days before computer solutions of the Navier-Stokes equations.

The aerofoil solution will not be presented here due to the mathematical complexity of the transformation, but the sink vortex and cylindrical obstruction are both reasonably tractable and will be developed in full.

Model 3: Sink vortex

A sink vortex is developed by superposing irrotational radial and circumferential flow, neither of which is entirely realistic in normal circumstances. Due to circular symmetry the solution is more easily developed in polar coordinates:

a) Simple radial flow with sink



Let Q = the total flow to the sink per unit depth

Define $Q < 0$ for a sink, $Q > 0$ for a source.

$$Q = 2\pi r V_r$$

V_r is the radial velocity, negative for a sink. This relates directly to the potential function:

$$V_r = Q/2\pi r = \frac{\partial \phi}{\partial r}$$

Given $V_r < 0$, ϕ must decrease with increasing r . The minimum radius at the sink is $r = c$.

Integrating for ϕ ,

$$\partial \phi = \frac{Q \partial r}{2\pi r}$$

$$\phi = \frac{Q}{2\pi} \ln\left(\frac{r}{c}\right)$$

The minimum radius c is handled as a logarithmic integration constant, thus
 $\phi = 0$ @ $r = c$, $\phi < 0$ for $r > c$.

The radial velocity also relates to the streamline function in the lateral direction. In polar coordinates the lateral dimension depends on the radius and an angle:

$$\partial w = r \partial \theta$$

$$V_r = \frac{\partial \psi}{\partial w} = \frac{\partial \psi}{r \partial \theta} = \frac{Q}{2\pi r}$$

Integrating for ψ ,

$$\partial \psi = \frac{Q \partial \theta}{2\pi}$$

$$\psi = \frac{Q \theta}{2\pi}$$

With the linear integration constant set to zero we obtain $\psi = 0$ @ $\theta = 0$, $\psi < 0$ for $\theta > 0$.

The potential and stream functions can be inverted for r and θ as functions of free parameters ψ and ϕ :

$$r = ce^{2\pi\phi/Q}$$

$$\theta = \frac{2\pi\psi}{Q}$$

which tends to explain the flow pattern illustrated above: the angular streamline steps are linear, while the radial potential steps are exponentially non-linear.

Note again that a radial *source* flow has $Q > 0$ so ψ and ϕ increase with increasing θ and r . It can be demonstrated experimentally that a purely radial sink is unstable, developing spontaneously into a vortex, while source flows are stable and will flow outward radially.

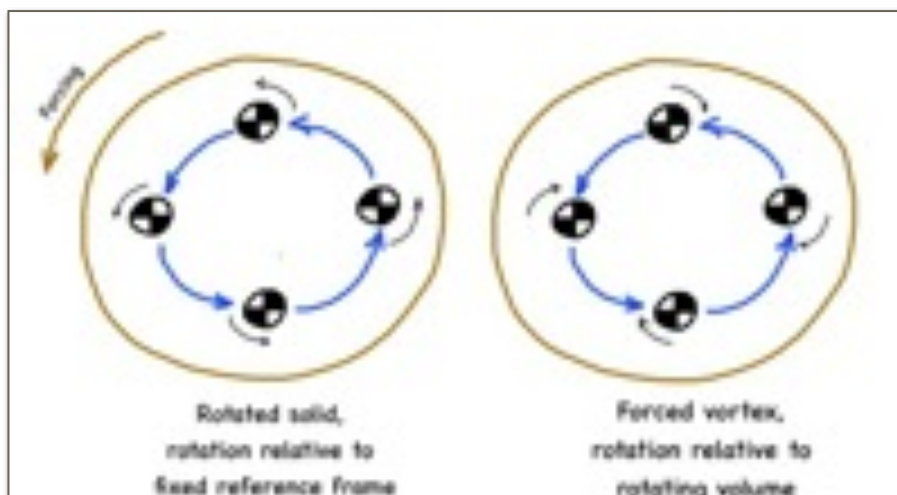
Note also that sink flow follows gravity towards a lower *potential* energy. The potential flow develops an irrotational pattern largely because of the uniform forcing mechanism of gravity. The action of potential energy thus partly explains the term "potential flow".

A sink vortex can be modelled by superposing a free vortex, also operating under gravitational forcing, on the radial flow.

b) Vortex flow; forced and free vortices

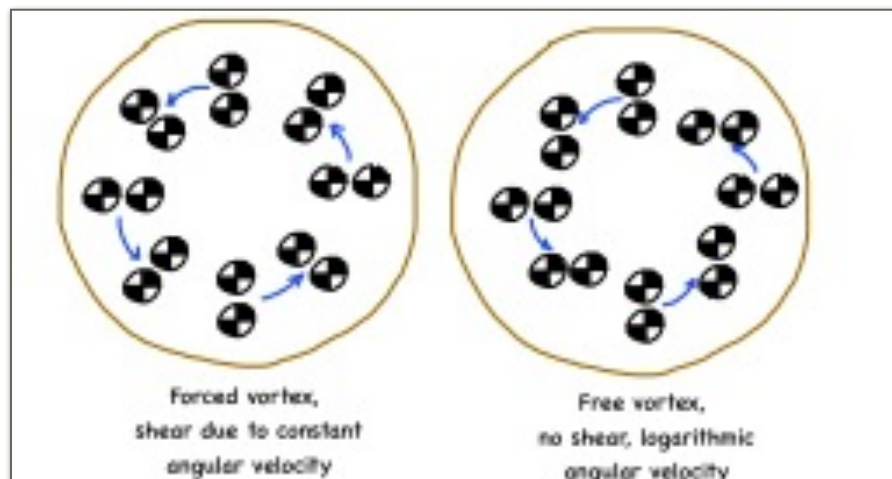
A *forced vortex* is a volume of fluid rotated on its axis by an external forcing mechanism, such as a bucket of fluid on a turntable or stirred by a vane. The vortex develops from a stationary volume to a forced rotating volume through complex intermediate stages.

A forced vortex is rotational, with non-zero curl. Fluid elements within the rotated volume tend to remain stationary relative to the fixed reference frame; relative to the rotated volume the fluid elements are actually in rotation. This generates shear stresses within the volume and cannot be modelled by potentiometric methods. Comparison with a rotated solid is useful:



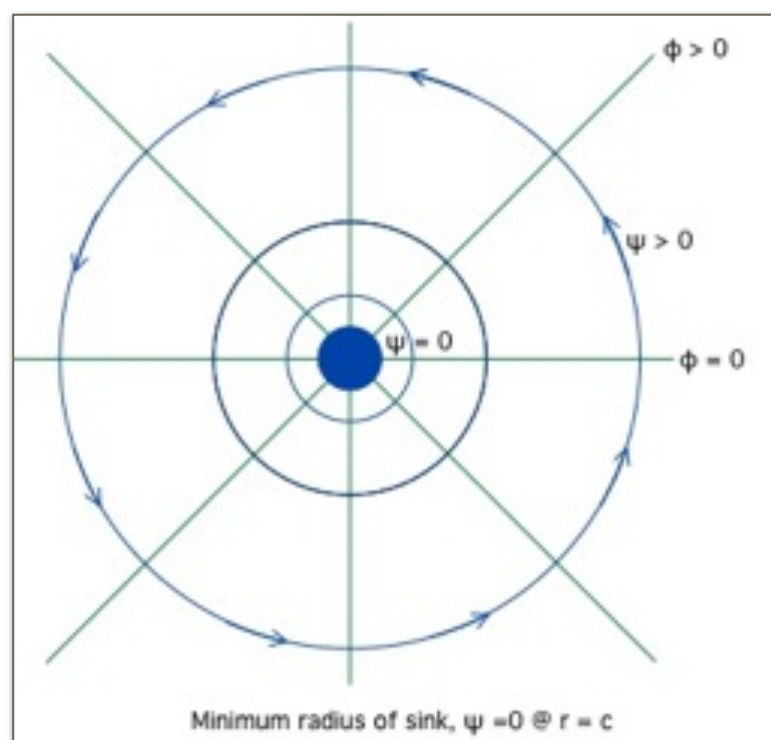
To visualise the kind of rotation depicted in the + fluid elements, try turning on the spot with a cup of coffee (slowly).

A *free vortex* in contrast is an irrotational vortex that develops after forcing has ended and the dominant force is again gravity. In a real free vortex, viscosity erodes kinetic energy and reduces relative velocity gradients to near zero. The ideal fluid flow is modelled from this endpoint, with no velocity gradients and therefore no relative rotation.



A key determinant of a free vortex is that there is *no relative difference in total energy or head across the flow, because there is no friction loss, so in the radial direction the Bernoulli equation is constant*. This also applies to radial sink flow and is consistent with the behaviour of real sink vortices.

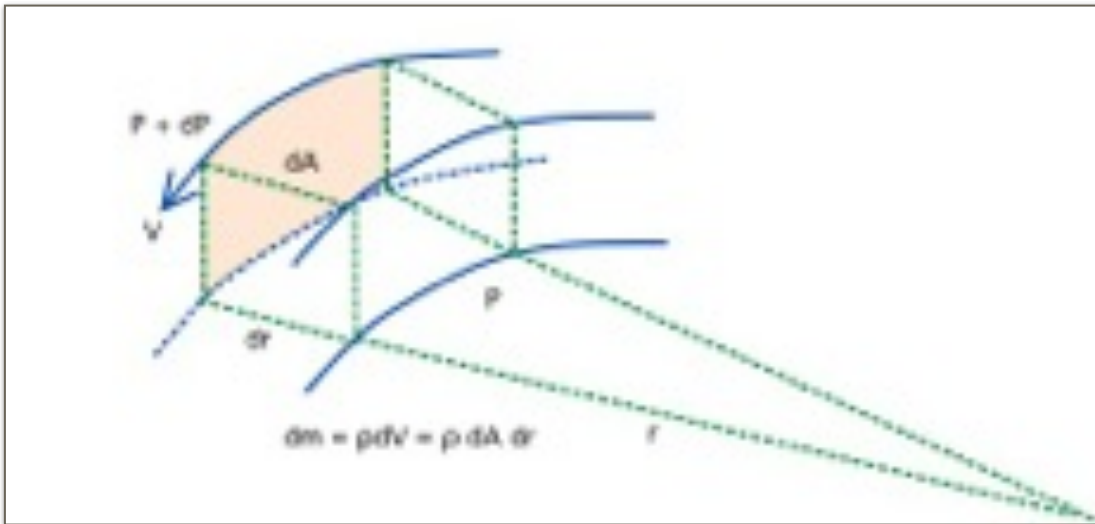
Assuming a similar construction as for a radial sink, with potential and streamlines interchanged:



The curved flow is an angular acceleration, ie a change in tangential velocity V , which is driven by a pressure gradient across the flow:

$$dF = dm \frac{\partial V}{\partial t} = dm \frac{V^2}{r}$$

$$\frac{\partial P}{\partial r} = \frac{\partial F}{\partial A \partial r} = \frac{\partial m}{\partial \forall} \frac{V^2}{r} = \rho \frac{V^2}{r}$$



The pressure gradient is used to evaluate Bernoulli's equation across the flow:

$$H = \frac{V^2}{2g} + \frac{P}{\rho g}$$

$$\frac{\partial H}{\partial r} = \frac{V \partial V}{g \partial r} + \frac{\partial P}{\partial r} \frac{1}{\rho g}$$

$$= \frac{V \partial V}{g \partial r} + \frac{V^2}{r} \frac{1}{g} \quad \text{Assuming no head difference across the flow (potential flow):}$$

$$\frac{\partial H}{\partial r} = 0$$

$$\frac{V \partial V}{g \partial r} + \frac{V^2}{r} \frac{1}{g} = 0$$

$$\frac{\partial V}{\partial r} = -\frac{V}{r}$$

This is a classic logarithmic differential equation. The solution is

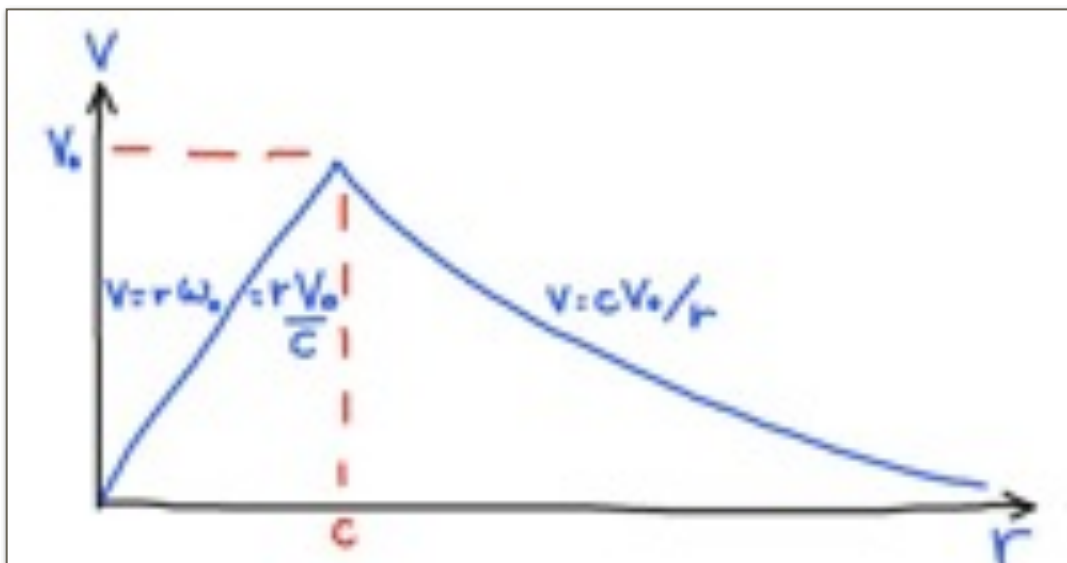
$$\frac{\partial V}{V} = -\frac{\partial r}{r}$$

$$\ln\left(\frac{V}{V_0}\right) = -\ln\left(\frac{r}{c}\right) = \ln\left(\frac{c}{r}\right)$$

$$V = \frac{cV_0}{r}$$

The minimum radius c requires some interpretation: A forced vortex develops into a free vortex from the outside inwards as wall friction erodes velocities. The forced vortex decays by decreasing in radius, but is never wholly eliminated as the overall vortex motion decays to a static state.

The minimum radius c is used to represent the transition from a free to a forced vortex at this quasi-end state, and can be set to coincide with the minimum radius of a sink. Inside the minimum radius the vortex has constant angular velocity ω_0 .



Note that there is no maximum radius, hence this model does not accurately reflect a confined forced-free vortex. When combined with a radial sink however, the model is a realistic model of an unconfined sink vortex.

Note that V is a tangential velocity. As will be seen in the graphical results of this model, V_0 must be negative for a right handed sink vortex.

The tangential velocity V is the stream function derivative in the radial direction:

$$V = \frac{\partial \psi}{\partial r} = \frac{cV_0}{r}$$

$$\partial \psi = \frac{cV_0}{r} \partial r$$

$$\psi = cV_0 \ln \left(\frac{r}{c} \right)$$

and also to the potential function in the angular direction:

$$V = \frac{\partial \phi}{r \partial \theta} = \frac{cV_0}{r}$$

$$\partial \phi = cV_0 \partial \theta$$

$$\phi = cV_0 \theta$$

As before these functions can be inverted for r and θ as functions of free parameters ψ and ϕ :

$$r = ce^{(\psi/cV_0)}$$

$$\theta = \phi/cV_0$$

Again, this tends to explain the flow pattern illustrated above: linear potential steps in the angular direction, and exponentially non-linear radial streamline steps.

c) Sink vortex: superposition of radial and vortex flows

The streamlines and potential functions for the radial and vortex flows can now be added together:

$$\psi = cV_0 \ln \left(\frac{r}{c} \right) + \frac{Q\theta}{2\pi}$$

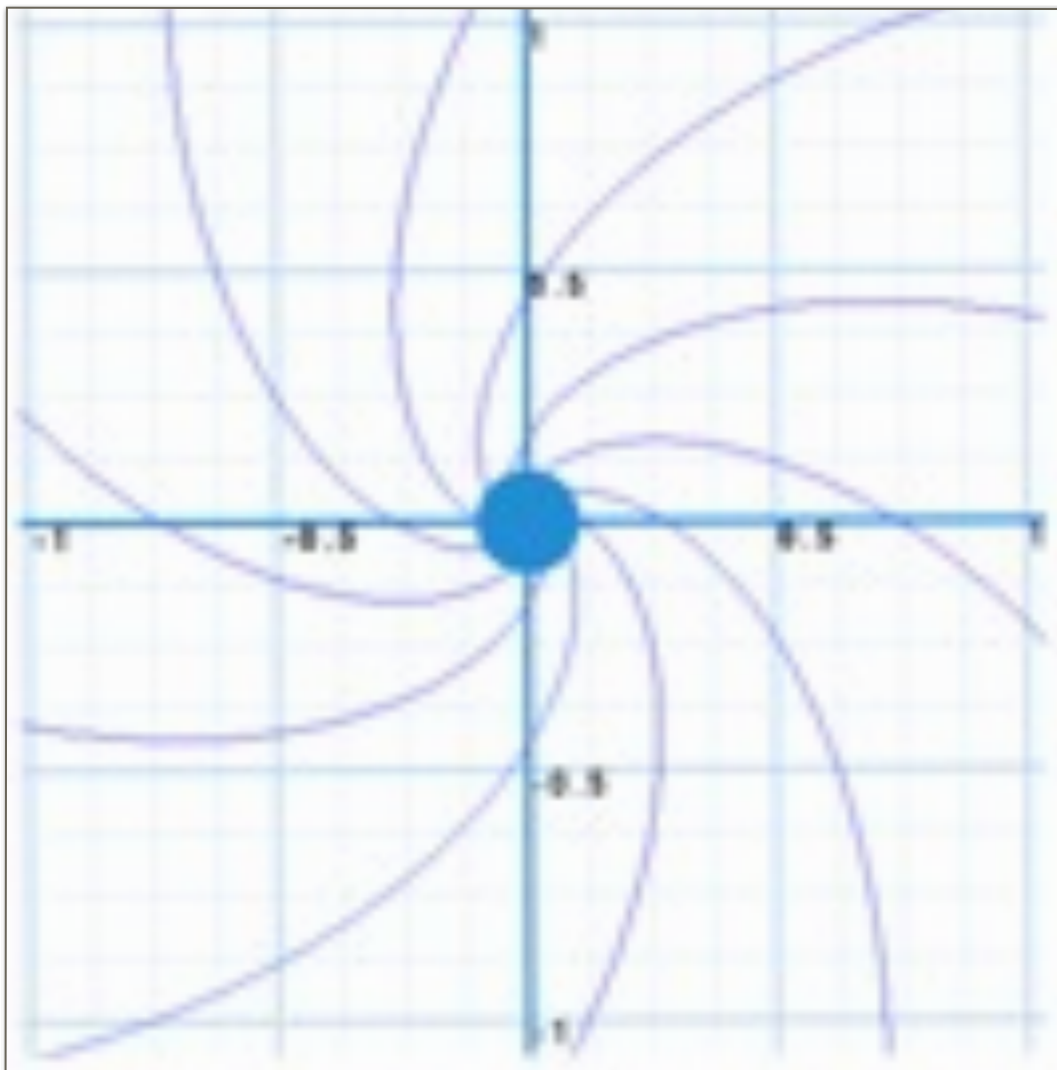
$$\phi = cV_0 \theta + \frac{Q}{2\pi} \ln \left(\frac{r}{c} \right)$$

The combined sink vortex is thus composed of similar exponential-linear curves with complementary coefficients and exponents. These can be plotted by inverting for r as a function of θ , ψ and ϕ :

$$r = ce^{(\psi - Q\theta/2\pi)/cV_0} \quad \text{streamlines}$$

$$r = ce^{2\pi(\phi - cV_0\theta)/Q} \quad \text{potential lines}$$

These polar functions are *logarithmic spirals* with opposite and complementary winding. Plotting the streamlines with $dQ = d\psi = 0.01$ for the parameters $c = 0.1$, $Q = V_0 = -0.1$ we obtain a fairly natural-looking sink vortex with $Q/dQ = 10$ distinct flow bands:



Further parameterisation based on experimental data is needed to find a natural relationship between Q and V_0 . The spirals shown above have a fairly tight winding which would correspond to the kinematic viscosity of a gas rather than a liquid.

Note that the velocity V_0 and flowrate Q are set as negative for a right-handed sink.

The velocity distribution of a sink vortex is also a linear sum of the radial sink and free vortex velocities. The radial and tangential velocities of the two models are independent, so the combined polar velocity distribution is;

$$V = (V_r, V_\theta) = \left(\frac{Q}{2\pi r}, \frac{cV_o}{r} \right)$$

These can be plotted as a pair of surfaces having negative values throughout:



The actual water surface elevation can be deduced from Bernoulli's equation:

$$\text{constant } H = \frac{V^2}{2g} + \frac{P}{\rho g} = \frac{V^2}{2g} + z$$

$$z = H - \frac{V^2}{2g}$$

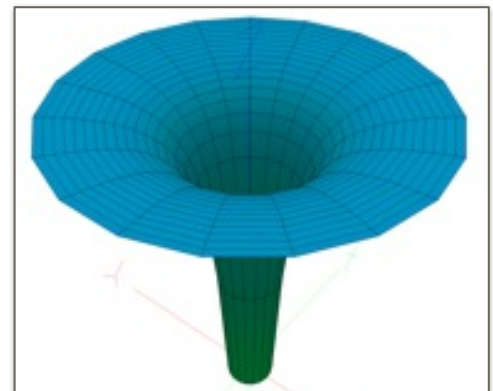
where z is the water surface elevation. Since V is a sum of independent components, V² is easy:

$$z = H - \frac{V^2}{2g} = H - \frac{\left(\left(\frac{Q}{2\pi r}\right)^2 + \left(\frac{cV_o}{r}\right)^2\right)}{2g} = H - \frac{\left(\left(\frac{Q}{2\pi}\right)^2 + (cV_o)^2\right)}{2gr^2}$$

The constant H is the initial value at r = c, assuming an initial surface elevation z_o:

$$H = z_o + \frac{\left(\left(\frac{Q}{2\pi}\right)^2 + (cV_o)^2\right)}{2gc^2}$$

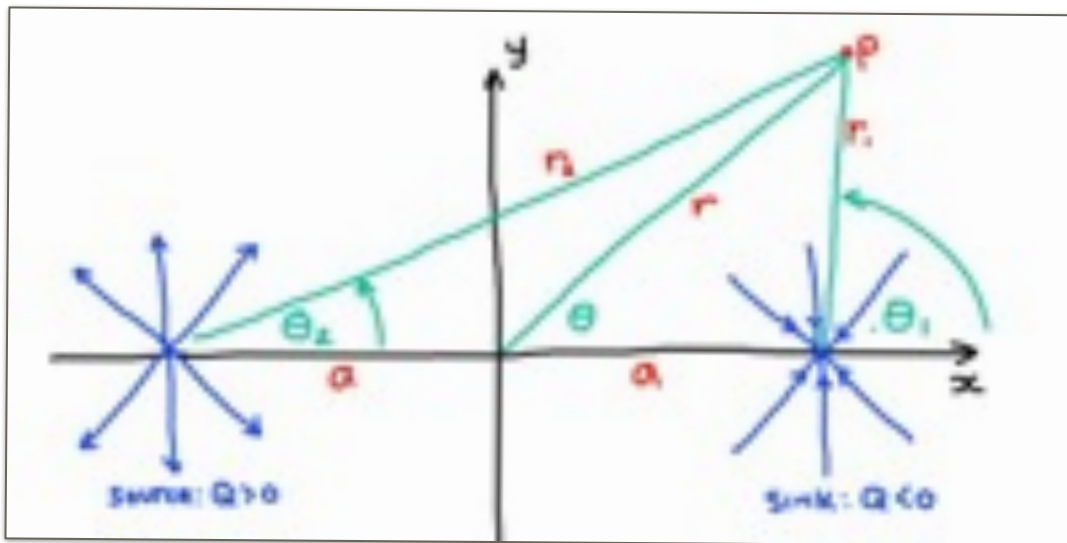
The resulting plot of elevation has its datum at the minimum radius and initial surface elevation. The steep vortex orifice is familiar from your bathtub.



Model 4: Laminar flow around a cylindrical obstruction

Paired radial flows, one a source and the other a sink, create what is known as a *doublet*; when combined with a uniform horizontal flow the doublet reproduces the irrotational streamline patterns around an ovoid or cylindrical obstruction. This was useful in early studies of laminar drag such as Stoke's law.

This solution is realistic only at very low velocities. well within the laminar regime. Any level of turbulence will result in the formation of wake vortices, usually in paired alternating eddies of positive and negative curl.



Definition of a point p in relation to a paired source-sink or doublet. Note that θ_2 is always less than θ_1 .

The streamline functions for the sink and source are added together (linear superposition):

$$\psi(\text{source}) = \frac{Q\theta_2}{2\pi}, \quad \psi(\text{sink}) = -\frac{Q\theta_1}{2\pi}$$

$$\psi = \frac{Q\theta_2}{2\pi} - \frac{Q\theta_1}{2\pi} = -\frac{Q}{2\pi}(\theta_1 - \theta_2)$$

This indicates that $\psi < 0$ as $\theta_2 < \theta_1$.

For a given semi-major axis a and point $P = (x, y)$, the radii and angles can be found by:

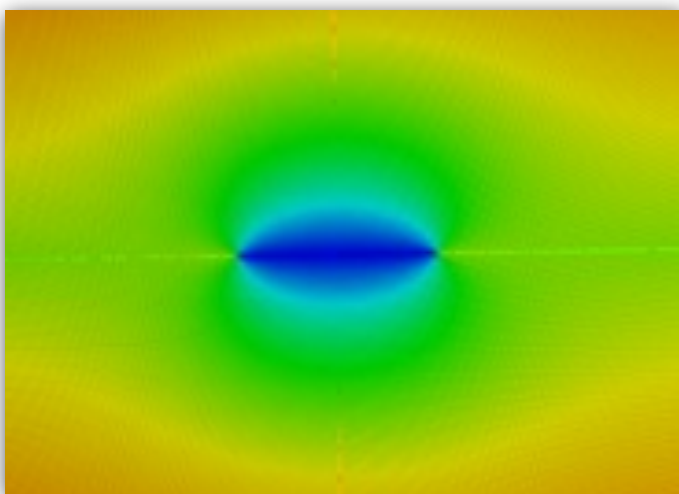
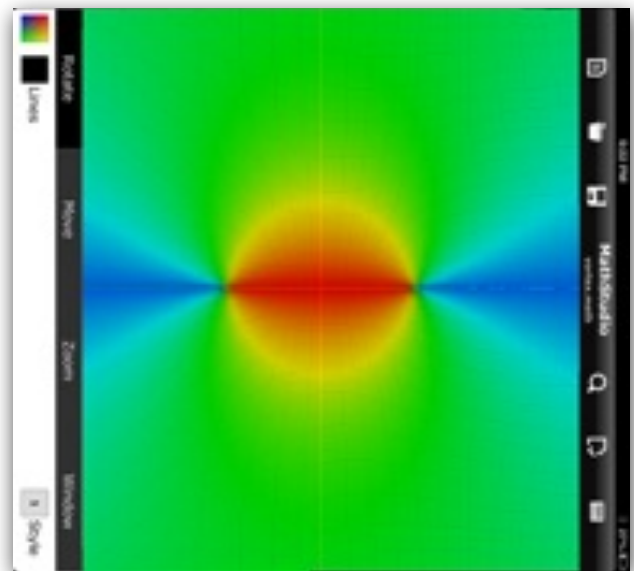
$$r_1 = \sqrt{(x - a)^2 + y^2}$$

$$r_2 = \sqrt{(x + a)^2 + y^2}$$

$$\theta_1 = \arccos\left(\frac{x - a}{r_1}\right)$$

$$\theta_2 = \arccos\left(\frac{x + a}{r_2}\right)$$

This associates a value of ψ for any point (x, y) . It would be difficult to invert this for x, y as a function of ψ . Instead ψ can be plotted directly from the above equations as a surface function of (x, y) with colours indicating contours of equal stream value:



A horizontal uniform flow is now added. For zero vertical flow, horizontal velocity = U , the streamline of uniform flow is simply $\psi = Uy$. Adding this additional component into the stream function we obtain a graphical solution:

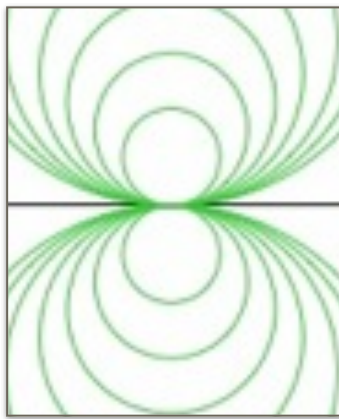
This is an outline of the flow around an ovoid cylindrical obstruction. The surface of the cylinder is a streamline and as such is not distinct in this view.

A circular cylindrical obstruction can be found by reducing the semi-major axis a to zero. At this point the solution with no horizontal flow simplifies to:

$$\psi = \frac{-k \sin(\theta)}{r}$$

$$\phi = \frac{k \cos(\theta)}{r}$$

Streamlines (lines of equal stream function) plot as circles tangential to the x axis:



Then with a uniform flow added in polar coordinates:

$$\psi = \frac{-k \sin(\theta)}{r} + Uy = \frac{-k \sin(\theta)}{r} + Ur \sin(\theta)$$

$$= r \sin(\theta) \left(U - \frac{k}{r^2} \right)$$

But $\psi = 0$ at the radius R of the cylinder:

$$\psi(R) = R \sin(\theta) \left(U - \frac{k}{R^2} \right) = 0$$

Thus $k = UR^2$ regardless of θ . k is known as the strength of the doublet.

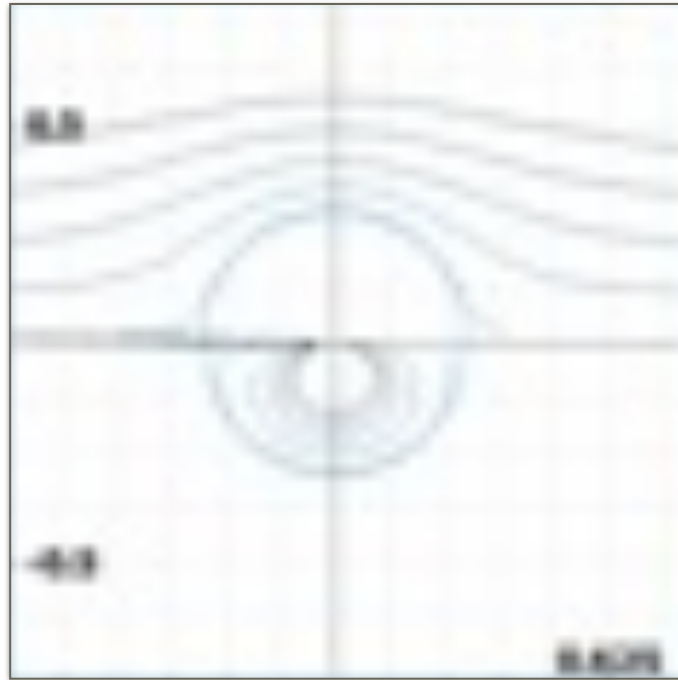
$$\psi = r \sin(\theta) \left(U - \frac{UR^2}{r^2} \right) = rU \sin(\theta) \left(1 - \frac{R^2}{r^2} \right)$$

The corresponding potential function is:

$$\phi = r \cos(\theta) \left(U + \frac{UR^2}{r^2} \right) = rU \cos(\theta) \left(1 + \frac{R^2}{r^2} \right)$$

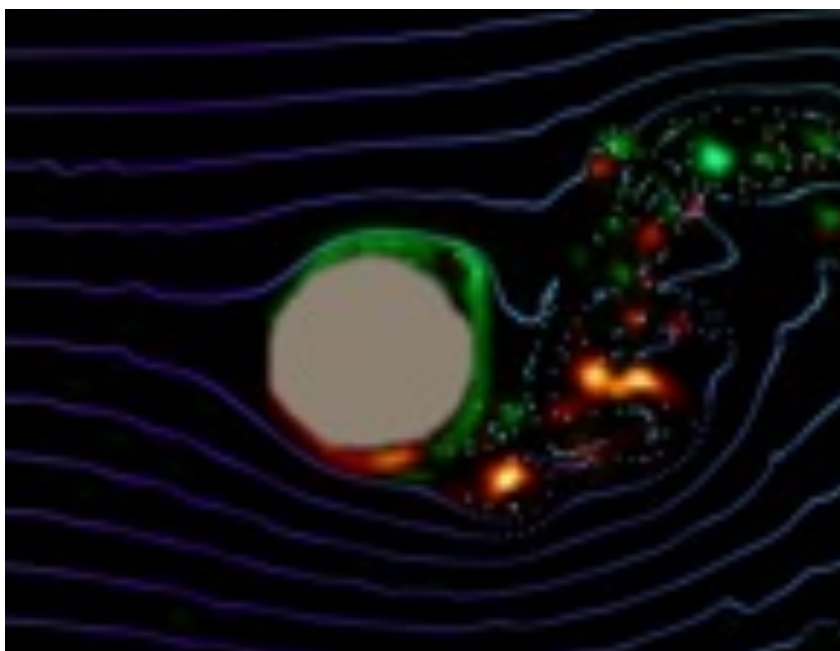
These solutions are realistic for $r \geq R$, ie $\theta \leq \pi$. The (quadratic) solution for r as a function of θ, ψ is:

$$r = \frac{\left(\frac{\psi}{U \sin(\theta)}\right) \pm \sqrt{\left(\frac{\psi}{U \sin(\theta)}\right)^2 + 4R^2}}{2}$$



This graphic shows the positive component of the quadratic solution, ie flow over one side of the cylinder. The circle indicates the cylinder's surface and is at $\psi = 0$. The ovoid traces inside the cylinder are non-realistic positive solutions for $\theta > \pi$.

The solution is not accurate at any natural or technological flowrate: a more technically correct solution, with viscosity, shows significant positive and negative vorticity.



See the next reading, Groundwater Flow Nets for graphical methods and other applications in groundwater modelling.