

Finite element method

Origins: structural mechanics, calculus of variations for elliptic BVPs

Boundary value problem

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ u = g_0 & \text{on } \Gamma_0 \\ \mathbf{n} \cdot \nabla u = g_1 & \text{on } \Gamma_1 \\ \mathbf{n} \cdot \nabla u + \alpha u = g_2 & \text{on } \Gamma_2 \end{cases}$$

$\begin{matrix} ? \\ \Leftrightarrow \end{matrix}$

Minimization problem

*Given a functional $J : V \rightarrow \mathbb{R}$
find $u \in V$ such that*

$$J(u) \leq J(w), \quad \forall w \in V$$

subject to the imposed BC

- $\oplus$ the functional contains derivatives of lower order
- $\oplus$ solutions from a broader class of functions are admissible
- $\oplus$ boundary conditions for complex domains can be handled easily
- $\ominus$ sometimes there is no functional associated with the original BVP

Modern FEM: weighted residuals formulation (weak form of the PDE)

Theory: 1D minimization problem

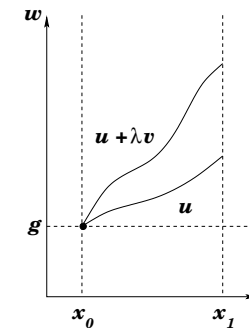
Minimize $J(w) = \int_{x_0}^{x_1} \psi \left(x, w, \frac{dw}{dx} \right) dx$ over $w \in V = C^2([x_0, x_1])$

subject to the boundary condition $w(x_0) = g_0$

Find $u \in V$ such that $J(u) \leq J(w)$ for all admissible w

$$w(x) = u(x) + \lambda v(x), \quad \lambda \in \mathbb{R}, \quad v \in V_0 = \{v \in V : v(x_0) = 0\}$$

$$\Rightarrow J(u + \lambda v) = \int_{x_0}^{x_1} \psi \left(x, u + \lambda v, \frac{du}{dx} + \lambda \frac{dv}{dx} \right) dx = I(\lambda)$$



By construction $w(x) = u(x)$ for $\lambda = 0$ so that

$$I(0) \leq I(\lambda), \quad \forall \lambda \in \mathbb{R} \quad \Rightarrow \quad \left. \frac{dI}{d\lambda} \right|_{\lambda=0} = 0, \quad \forall v \in V_0$$

Necessary condition of an extremum

$$\delta J(u, v) \stackrel{\text{def}}{=} \left. \frac{d}{d\lambda} J(u + \lambda v) \right|_{\lambda=0} = 0, \quad \forall v \in V_0 \quad \text{the first variation of the functional must vanish}$$

Necessary condition of an extremum

Chain rule for the function $\psi = \psi(x, w, w'), \quad w' = \frac{dw}{dx}$

$$\frac{d\psi}{d\lambda} = \frac{\partial\psi}{\partial x} \frac{dx}{d\lambda} + \frac{\partial\psi}{\partial w} \frac{dw}{d\lambda} + \frac{\partial\psi}{\partial w'} \frac{dw'}{d\lambda}, \quad \text{where} \quad \frac{dx}{d\lambda} = 0, \quad \frac{dw}{d\lambda} = v, \quad \frac{dw'}{d\lambda} = \frac{dv}{dx}$$

First variation of the functional

$$\left. \frac{d}{d\lambda} J(u + \lambda v) \right|_{\lambda=0} = \int_{x_0}^{x_1} \left[\frac{\partial\psi}{\partial w} v + \frac{\partial\psi}{\partial w'} \frac{dv}{dx} \right] dx = 0, \quad \forall v \in V_0$$

Integration by parts using the boundary condition $v(x_0) = 0$ yields

$$\int_{x_0}^{x_1} \left[\frac{\partial\psi}{\partial w} - \frac{d}{dx} \left(\frac{\partial\psi}{\partial w'} \right) \right] v dx + \frac{\partial\psi}{\partial w'}(x_1) v(x_1) = 0, \quad \forall v \in V_0$$

including $\forall v \in \hat{V} = \{v \in V_0 : v(x_1) = 0\} \Rightarrow \frac{\partial\psi}{\partial w} - \frac{d}{dx} \left(\frac{\partial\psi}{\partial w'} \right) = 0$

Substitution $\Rightarrow \frac{\partial\psi}{\partial w'}(x_1) \underbrace{v(x_1)}_{\text{arbitrary}} = 0, \quad \forall v \in V_0 \Rightarrow \frac{\partial\psi}{\partial w'}(x_1) = 0$

Du Bois Reymond lemma

LEMMA. Let $f \in C([a, b])$ be a continuous function and assume that

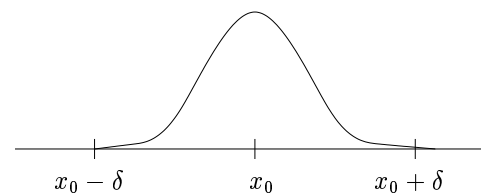
$$\int_a^b f(x)v(x) dx = 0, \quad \forall v \in \hat{V} = \{v \in C^2([a, b]) : v(a) = v(b) = 0\}$$

Then $f(x) = 0, \quad \forall x \in [a, b]$.

PROOF. Suppose $\exists x_0 \in (a, b)$ such that $f(x_0) \neq 0$, e.g. $f(x_0) > 0$

f is continuous $\Rightarrow f(x) > 0, \quad x \in (x_0 - \delta, x_0 + \delta) \subset (a, b)$

$$\text{Let } v(x) = \begin{cases} \exp\left(-\frac{1}{\delta^2 - (x - x_0)^2}\right) & \text{if } |x - x_0| < \delta \\ 0 & \text{if } |x - x_0| \geq \delta \end{cases}$$



$$v \in \hat{V} \quad \text{but} \quad \int_a^b f(x)v(x) dx = \int_{x_0 - \delta}^{x_0 + \delta} f(x) \exp\left(-\frac{1}{\delta^2 - (x - x_0)^2}\right) dx > 0$$

$$\Rightarrow f(x) = 0, \quad \forall x \in (a, b) \quad f \in C([a, b]) \quad \Rightarrow f \equiv 0 \quad \text{in } [a, b] \quad \square$$

Example: 1D Poisson equation

Constraints imposed on the solution $w = u$ of the minimization problem

$$\begin{cases} \frac{\partial \psi}{\partial w} - \frac{d}{dx} \left(\frac{\partial \psi}{\partial w'} \right) = 0 & \text{Euler-Lagrange equation} \\ u(x_0) = g_0 & \text{essential boundary condition} \\ \frac{\partial \psi}{\partial w'}(x_1) = 0 & \text{natural boundary condition} \end{cases}$$

Poisson equation: the solution $u \in V_g = \{v \in C^2([0, 1]) : v(0) = g_0\}$

minimizes the functional $J(w) = \int_0^1 \left[\frac{1}{2} \left(\frac{dw}{dx} \right)^2 - fw \right] dx, \quad w \in V_g$

$$\begin{aligned} \psi(x, w, w') &= \frac{1}{2} \left(\frac{dw}{dx} \right)^2 - fw, & \frac{\partial \psi}{\partial w} &= -f \\ \frac{\partial \psi}{\partial w'} &= \frac{dw}{dx}, & \frac{d}{dx} \left(\frac{\partial \psi}{\partial w'} \right) &= \frac{d^2 w}{dx^2} \end{aligned} \quad \begin{cases} -\frac{d^2 u}{dx^2} = f & \text{in } (0, 1) \\ u(0) = g_0 & \text{Dirichlet BC} \\ \frac{du}{dx}(1) = 0 & \text{Neumann BC} \end{cases}$$

Example: 2D Poisson equation

Find $u \in V_g = \{v \in C^2(\Omega) \cap C(\bar{\Omega}) : v|_{\Gamma_0} = g_0\}$ that minimizes the functional

$$J(w) = \int_{\Omega} \left[\frac{1}{2} |\nabla w|^2 - fw \right] d\mathbf{x} - \int_{\Gamma_1} g_1 w ds - \int_{\Gamma_2} g_2 w ds + \frac{\alpha}{2} \int_{\Gamma_2} w^2 ds, \quad w \in V_g$$

where $f = f(\mathbf{x})$, $g_0 = g_0(\mathbf{x})$, $g_1 = g_1(\mathbf{x})$, $g_2 = g_2(\mathbf{x})$, $\alpha \geq 0$

Admissible functions $w = u + \lambda v$, $v \in V_0 = \{v \in C^2(\Omega) \cap C(\bar{\Omega}) : v|_{\Gamma_0} = 0\}$

Necessary condition of an extremum $\frac{d}{d\lambda} J(u + \lambda v)|_{\lambda=0} = 0$, $\forall v \in V_0$

$$\frac{\partial}{\partial \lambda} |\nabla w|^2 = \frac{\partial}{\partial \lambda} \left[\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right] = 2 \left[\frac{\partial w}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial v}{\partial y} \right] = 2 \nabla w \cdot \nabla v$$

$$\int_{\Omega} [\nabla u \cdot \nabla v - fv] d\mathbf{x} - \int_{\Gamma_1} g_1 v ds - \int_{\Gamma_2} g_2 v ds + \alpha \int_{\Gamma_2} uv ds = 0, \quad \forall v \in V_0$$

Example: 2D Poisson equation

Integration by parts using Green's formula

$$\int_{\Omega} [-\Delta u - f]v \, d\mathbf{x} + \int_{\Gamma_1 \cup \Gamma_2} (\mathbf{n} \cdot \nabla u)v \, ds - \int_{\Gamma_1} g_1 v \, ds - \int_{\Gamma_2} g_2 v \, ds + \alpha \int_{\Gamma_2} uv \, ds = 0$$

$$\forall v \in V_0 \text{ including } v \in C^2(\Omega) \cap C(\bar{\Omega}) : v|_{\Gamma} = 0 \quad \Rightarrow \quad \int_{\Omega} [-\Delta u - f]v \, d\mathbf{x} = 0$$

Du Bois Reymond lemma: $-\Delta u = f$ in Ω Euler-Lagrange equation

$$\int_{\Gamma_1} [\mathbf{n} \cdot \nabla u - g_1]v \, ds + \int_{\Gamma_2} [\mathbf{n} \cdot \nabla u + \alpha u - g_2]v \, ds = 0, \quad \forall v \in V_0$$

$$\text{Consider } v \in C^2(\Omega) \cap C(\bar{\Omega}) : v|_{\Gamma_0 \cup \Gamma_1} = 0 \quad \Rightarrow \quad \int_{\Gamma_2} = 0, \quad \mathbf{n} \cdot \nabla u + \alpha u = g_2$$

Substitution yields $\int_{\Gamma_1} = 0$, $\mathbf{n} \cdot \nabla u = g_1$ and the following BVP

$$\left\{ \begin{array}{lll} -\Delta u = f & \text{in } \Omega & \text{2D Poisson equation} \\ u = g_0 & \text{on } \Gamma_0 & \text{Dirichlet BC (essential)} \\ \mathbf{n} \cdot \nabla u = g_1 & \text{on } \Gamma_1 & \text{Neumann BC (natural)} \\ \mathbf{n} \cdot \nabla u + \alpha u = g_2 & \text{on } \Gamma_2 & \text{Robin BC (natural)} \end{array} \right.$$

Rayleigh-Ritz method

Exact solution

$$u \in V : \quad u = \varphi_0 + \sum_{j=1}^{\infty} c_j \varphi_j$$

Approximate solution

$$u_h \in V_h : \quad u_h = \varphi_0 + \sum_{j=1}^N c_j \varphi_j$$

φ_0 an arbitrary function satisfying $\varphi_0 = g_0$ on Γ_0

φ_j basis functions vanishing on the boundary part Γ_0

Continuous problem

Find $u \in V$ such that
 $J(u) \leq J(w), \quad \forall w \in V$

Discrete problem

Find $u_h \in V_h$ such that
 $J(u_h) \leq J(w_h), \quad \forall w_h \in V_h$

$$J(c_1, \dots, c_N) = \min_{w_h \in V_h} J(w_h) \quad \Rightarrow \quad \frac{\partial J}{\partial c_i} = 0, \quad i = 1, \dots, N$$

Linear system: $Ac = b, \quad A \in \mathbb{R}^{N \times N}, \quad b \in \mathbb{R}^N, \quad c = [c_1, \dots, c_N]^T$

Example: 1D Poisson equation

Find the coefficients $c_1, \dots, c_N$ that minimize the functional

$$J(w_h) = \int_0^1 \left[\frac{1}{2} \left(\frac{dw_h}{dx} \right)^2 - f w_h \right] dx, \quad w_h = \sum_{j=1}^N c_j \varphi_j$$

Necessary condition of an extremum

$$\frac{\partial J}{\partial c_i} = \frac{\partial}{\partial c_i} \left[\frac{1}{2} \int_0^1 \left(\sum_{j=1}^N c_j \frac{d\varphi_j}{dx} \right)^2 dx - \int_0^1 f \left(\sum_{j=1}^N c_j \varphi_j \right) dx \right] = 0$$

$$\int_0^1 \frac{d\varphi_i}{dx} \left(\sum_{j=1}^N c_j \frac{d\varphi_j}{dx} \right) dx = \int_0^1 f \varphi_i dx, \quad i = 1, \dots, N$$

This is a linear system of the form $Ac = b$ with coefficients

$$a_{ij} = \int_0^1 \frac{d\varphi_i}{dx} \frac{d\varphi_j}{dx} dx, \quad b_i = \int_0^1 f \varphi_i dx, \quad c = [c_1, \dots, c_N]^T$$

Example: poor choice of the basis functions

Consider the polynomial basis $\varphi_0 = 0, \quad \varphi_i = x^i, \quad i = 1, \dots, N$

$$u_h(x) = \sum_{j=1}^N c_j x^j, \quad a_{ij} = \int_0^1 ij x^{i+j-2} dx = \frac{ij}{i+j-1}, \quad b_i = \int_0^1 f x^i dx$$

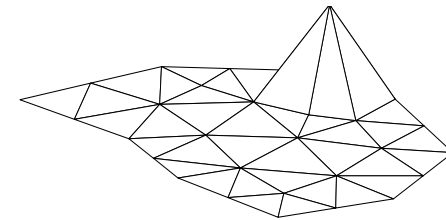
$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdot & 1 \\ 1 & 4/3 & 6/4 & 8/5 & \cdot & 2N/(N+1) \\ 1 & 6/4 & 9/5 & 12/6 & \cdot & 3N/(N+2) \\ 1 & 8/5 & 12/6 & 16/7 & \cdot & 4N/(N+3) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & N^2/(2N-1) \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ \cdot \\ b_N \end{bmatrix}, \quad c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ \cdot \\ c_N \end{bmatrix}$$

- A is known as the Hilbert matrix which is SPD but full and ill-conditioned so that the solution is expensive and corrupted by round-off errors.
- for A to be sparse, the basis functions should have a compact support

Fundamentals of the FEM

The *Finite Element Method* is a systematic approach to generating piecewise-polynomial basis functions with favorable properties

- The computational domain Ω is subdivided into many small subdomains K called *elements*: $\bar{\Omega} = \bigcup_{K \in \mathcal{T}_h} K$.
- The *triangulation* $\mathcal{T}_h$ is admissible if the intersection of any two elements is either an empty set or a common vertex / edge / face of the mesh.
- The finite element subspace V_h consists of piecewise-polynomial functions. Typically, $V_h = \{v \in C^m(\Omega) : v|_K \in P_k, \forall K \in \mathcal{T}_h\}$.
- Any function $v \in V_h$ is uniquely determined by a finite number of *degrees of freedom* (function values or derivatives at certain points called *nodes*).
- Each basis function φ_i accommodates exactly one degree of freedom and has a small support so that the resulting matrices are sparse.



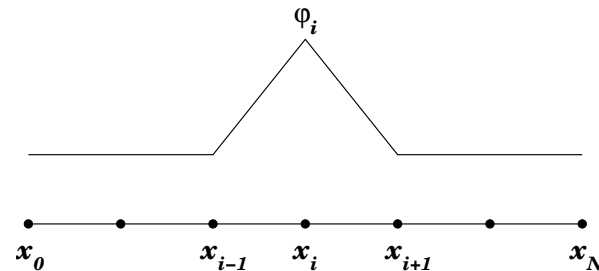
Finite element approximation

The *finite element* is a triple (K, P, Σ) , where

- K is a closed subset of $\bar{\Omega}$
- P is the polynomial space for the shape functions
- Σ is the set of local degrees of freedom

Basis functions possess the property

$$\varphi_j(x_i) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$



$$u_h(x) = \sum_{j=1}^N u_j \varphi_j(x) \quad \Rightarrow \quad u_h(x_i) = \sum_{j=1}^N u_j \varphi_j(x_i) = \sum_{j=1}^N u_j \delta_{ij} = u_i$$

Approximate solution: the nodal values $u_1, \dots, u_N$ can be computed by the Ritz method provided that there exists an equivalent minimization problem

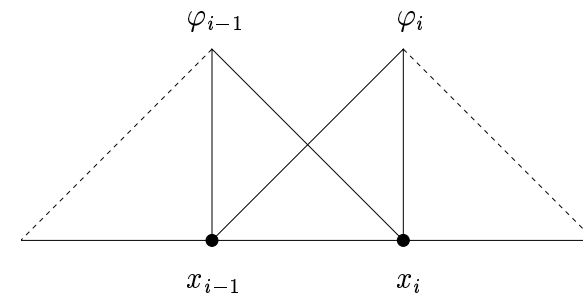
Example: 1D Poisson equation

Find the nodal values $u_1, \dots, u_N$ that minimize the functional

$$J(w_h) = \int_0^1 \left[\frac{1}{2} \left(\frac{dw_h}{dx} \right)^2 - f w_h \right] dx, \quad w_h = \sum_{j=1}^N u_j \varphi_j$$

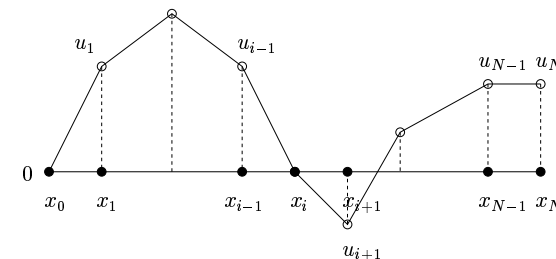
Local basis functions for $e_i = [x_{i-1}, x_i]$

$$\varphi_{i-1}(x) = \frac{x_i - x}{x_i - x_{i-1}}, \quad \varphi_i(x) = \frac{x - x_{i-1}}{x_i - x_{i-1}}$$



Approximate solution for $x \in e_i$

$$\begin{aligned} u_h(x) &= \sum_{j=1}^N u_j \varphi_j = u_{i-1} \varphi_{i-1} + u_i \varphi_i \\ &= u_{i-1} + \frac{x - x_{i-1}}{x_i - x_{i-1}} (u_i - u_{i-1}) \end{aligned}$$



continuous, piecewise-linear

Example: 1D Poisson equation

The Ritz method yields a linear system of the form $Au = F$, where

$$a_{ij} = \sum_{k=1}^N \int_{e_k} \frac{d\varphi_i}{dx} \frac{d\varphi_j}{dx} dx, \quad F_i = \sum_{k=1}^N \int_{e_k} f \varphi_i dx$$

These integrals can be evaluated exactly or numerically (using a quadrature rule)

Stiffness matrix and *load vector* for a uniform mesh with $\Delta x = \frac{1}{N}$ and $f \equiv 1$

$$A = \frac{1}{(\Delta x)^2} \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \cdot & \cdot & \cdot & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 \\ 1 \\ \cdot \\ 1 \\ 1/2 \end{bmatrix}$$

This is the same linear system as the one obtained for the finite difference method!

Existence of a minimization problem

Sufficient conditions for an elliptic PDE $\mathcal{L}u = f$ in Ω , $u|_{\Gamma} = 0$

to be the Euler-Lagrange equation of a variational problem read

- the operator $\mathcal{L}$ is linear
- the operator $\mathcal{L}$ is self-adjoint (symmetric)

$$\int_{\Omega} v \mathcal{L}u \, dx = \int_{\Omega} u \mathcal{L}v \, dx \quad \text{for all admissible } u, v$$

- the operator $\mathcal{L}$ is positive definite

$$\int_{\Omega} u \mathcal{L}u \, dx \geq 0 \quad \text{for all admissible } u; \quad u \equiv 0 \quad \text{if } \int_{\Omega} u \mathcal{L}u \, dx = 0$$

In this case, the unique solution u minimizes the functional

$$J(w) = \frac{1}{2} \int_{\Omega} w \mathcal{L}w \, dx - \int_{\Omega} f w \, dx$$

over the set of admissible functions. Non-homogeneous BC modify this set and/or give rise to additional terms in the functional to be minimized

Example: 1D Poisson equation

Laplace operator $\mathcal{L} = -\frac{d^2}{dx^2}$ is linear and self-adjoint

$$\begin{aligned}\int_0^1 v \mathcal{L} u \, dx &= - \int_0^1 \frac{d^2 u}{dx^2} v \, dx = \int_0^1 \frac{du}{dx} \frac{dv}{dx} \, dx - \left[\frac{du}{dx} v \right]_0^1 \\ &= - \int_0^1 u \frac{d^2 v}{dx^2} \, dx + \left[u \frac{dv}{dx} \right]_0^1 = \int_0^1 u \mathcal{L} v \, dx\end{aligned}$$

Positive-definiteness: $\int_0^1 u \mathcal{L} u \, dx = - \int_0^1 u \frac{d^2 u}{dx^2} \, dx = \int_0^1 \left(\frac{du}{dx} \right)^2 \, dx \geq 0$

If $\int_0^1 \left(\frac{du}{dx} \right)^2 \, dx = 0$ then $\frac{du}{dx} \equiv 0 \Rightarrow u \equiv 0$ since $u(0) = 0$

Functional for the minimization problem

$$J(w) = \int_0^1 \left[\frac{1}{2} \left(\frac{dw}{dx} \right)^2 - fw \right] dx, \quad w(0) = 0$$

Non-homogeneous BC $u(0) = g_0 \longrightarrow w(0) = g_0$ (essential)

$\frac{du}{dx}(1) = g_1 \longrightarrow J(w) = \int_0^1 \left[\frac{1}{2} \left(\frac{dw}{dx} \right)^2 - fw \right] dx - g_1 w(1)$ (natural)

Least-squares method

Idea: minimize the *residual* of the PDE

$$R(w) = \mathcal{L}w - f \quad \text{such that} \quad R(u) = 0 \quad \Rightarrow \quad \mathcal{L}u = f$$

Least-squares functional $J(w) = \int_{\Omega} (\mathcal{L}w - f)^2 dx$ always exists

Necessary condition of an extremum

$$\left. \frac{d}{d\lambda} J(u + \lambda v) \right|_{\lambda=0} = \frac{d}{d\lambda} \left[\int_{\Omega} (\mathcal{L}(u + \lambda v) - f)^2 dx \right]_{\lambda=0} = 0$$

Integration by parts: $\int_{\Omega} (\mathcal{L}u - f) \mathcal{L}v dx = \int_{\Omega} \mathcal{L}^*(\mathcal{L}u - f) v dx - \int_{\Gamma} [\dots] ds = 0$

Euler-Lagrange equation $\mathcal{L}^* \mathcal{L}u = \mathcal{L}^* f$ where $\mathcal{L}^*$ is the *adjoint operator*

- corresponds to a derivative of the original PDE
- requires additional boundary conditions and extra smoothness
- it makes sense to rewrite a high-order PDE as a first-order system

Advantage: the matrices for a least-squares discretization are symmetric

Weighted residuals formulation

Idea: render the residual orthogonal to a space of test functions

Let $u = \sum_{j=1}^{\infty} \alpha_j \varphi_j \in V_0$ be the solution of
$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{cases}$$

Residual is zero if its projection onto each basis function equals zero

$$\mathcal{L}u - f = 0 \quad \Leftrightarrow \quad \int_{\Omega} (\mathcal{L}u - f) \varphi_i dx = 0 \quad \forall i = 1, 2, \dots$$

Test functions $v = \sum_{j=1}^{\infty} \beta_j \varphi_j \Rightarrow \int_{\Omega} (\mathcal{L}u - f) v dx = 0, \quad \forall v \in V_0$

Weak formulation: find $u \in V_0$ such that $a(u, v) = l(v) \quad \forall v \in V_0$

where $a(u, v) = \int_{\Omega} \mathcal{L}u v dx$ is a bilinear form and $l(v) = \int_{\Omega} f v dx$

Integration by parts: $\mathcal{L}u = \nabla \cdot \mathbf{g}(u) \Rightarrow a(u, v) = - \int_{\Omega} \mathbf{g}(u) \cdot \nabla v dx$

Finite element discretization

Continuous problem

$$\begin{aligned} &\text{Find } u \in V_0 \text{ such that} \\ &a(u, v) = l(v), \quad \forall v \in V_0 \end{aligned}$$

Discrete problem

$$\begin{aligned} &\text{Find } u_h \in V_h \subset V_0 \text{ such that} \\ &a(u_h, v_h) = l(v_h), \quad \forall v_h \in V'_h \end{aligned}$$

FEM approximations:

$$u_h = \sum_{j=1}^N u_j \varphi_j \in V_h, \quad v_h = \sum_{j=1}^N v_j \psi_j \in V'_h$$

where $V_h = \text{span}\{\varphi_1, \dots, \varphi_N\}$ and $V'_h = \text{span}\{\psi_1, \dots, \psi_N\}$ may differ

(Bubnov-)Galerkin method $V'_h = V_h \rightarrow a(u, \varphi_i) = l(\varphi_i), \quad \psi_i = \varphi_i$

Petrov-Galerkin method $V'_h \neq V_h \rightarrow a(u, \psi_i) = l(\psi_i), \quad \psi_i \neq \varphi_i$

Linear algebraic system

$$\sum_{j=1}^N a(\varphi_j, \psi_i) u_j = l(\psi_i), \quad \forall i = 1, \dots, N$$

Matrix form $Au = F$ with coefficients $a_{ij} = a(\varphi_j, \psi_i), \quad F_i = l(\psi_i)$

Example: 1D Poisson equation

Boundary value problem

$$\begin{cases} -\frac{d^2 u}{dx^2} = f & \text{in } (0, 1) \\ u(0) = 0, \quad \frac{du}{dx}(1) = 0 \end{cases}$$

Weak formulation

$$u \in V_0$$

$$\int_0^1 \left(-\frac{d^2 u}{dx^2} - f \right) v \, dx = 0, \quad \forall v \in V_0$$

Integration by parts yields

$$-\int_0^1 \frac{d^2 u}{dx^2} v \, dx = \int_0^1 \frac{du}{dx} \frac{dv}{dx} \, dx - \left[\frac{du}{dx} v \right]_0^1$$

Approximate solution

$$u_h(x) = \sum_{j=1}^N u_j \varphi_j(x)$$

Continuous problem $a(u, v) = l(v), \quad a(u, v) = \int_0^1 \frac{du}{dx} \frac{dv}{dx} \, dx, \quad l(v) = \int_0^1 f v \, dx$

Discrete problem $a(u_h, \varphi_i) = l(\varphi_i), \quad i = 1, \dots, N \quad (\text{Galerkin method})$

This is a (sparse) linear system of the form $Au = F$, where

$$a_{ij} = \int_0^1 \frac{d\varphi_i}{dx} \frac{d\varphi_j}{dx} \, dx, \quad F_i = \int_0^1 f \varphi_i \, dx, \quad u = [u_1, \dots, u_N]^T$$

The Galerkin and Ritz methods are equivalent if the minimization problem exists

Example: 2D Poisson equation

Boundary value problem

Weak formulation

$u \in V_g$

$$\left\{ \begin{array}{ll} -\Delta u = f & \text{in } \Omega \\ u = g_0 & \text{on } \Gamma_0 \\ \mathbf{n} \cdot \nabla u = g_1 & \text{on } \Gamma_1 \\ \mathbf{n} \cdot \nabla u + \alpha u = g_2 & \text{on } \Gamma_2 \end{array} \right.$$

$$\int_{\Omega} [-\Delta u - f]v \, d\mathbf{x} = 0, \quad \forall v \in V_0$$

$$V_g = \{v \in V : v|_{\Gamma_0} = g_0\}$$

$$V_0 = \{v \in V : v|_{\Gamma_0} = 0\}$$

Integration by parts using Green's formula

$$\int_{\Omega} \nabla u \cdot \nabla v \, d\mathbf{x} - \int_{\Gamma} (\mathbf{n} \cdot \nabla u)v \, ds = \int_{\Omega} f v \, d\mathbf{x}, \quad \Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$$

Boundary conditions $\int_{\Gamma_0} (\mathbf{n} \cdot \nabla u)v \, ds = 0$ since $v = 0$ on Γ_0

$$\int_{\Gamma_1} (\mathbf{n} \cdot \nabla u)v \, ds = \int_{\Gamma_1} g_1 v \, ds, \quad \int_{\Gamma_2} (\mathbf{n} \cdot \nabla u)v \, ds = \int_{\Gamma_2} g_2 v \, ds - \alpha \int_{\Gamma_2} u v \, ds$$

Approximate solution $u_h(\mathbf{x}) = \varphi_0 + \sum_{j=1}^N u_j \varphi_j(\mathbf{x}), \quad \varphi_0|_{\Gamma_0} = g_0$

Example: 2D Poisson equation

Continuous problem $a(u, v) = l(v) + \int_{\Gamma_1} g_1 v \, ds + \int_{\Gamma_2} g_2 v \, ds, \quad \forall v \in V_0$

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, d\mathbf{x} + \alpha \int_{\Gamma_2} uv \, ds, \quad l(v) = \int_{\Omega} f v \, d\mathbf{x}$$

Discrete problem $a(u_h, \varphi_i) = l(\varphi_i) + \int_{\Gamma_1} g_1 \varphi_i \, ds + \int_{\Gamma_2} g_2 \varphi_i \, ds, \quad \forall i = 1, \dots, N$

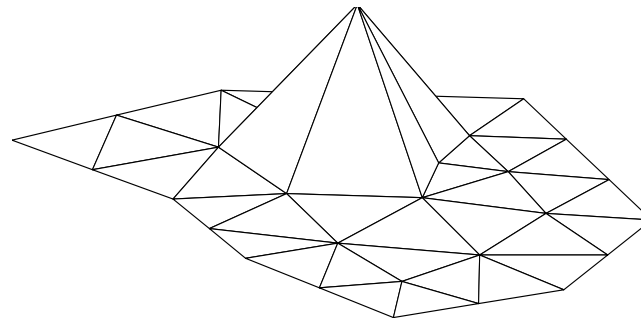
Piecewise-linear basis functions $\varphi_i \in C(\bar{\Omega}), \quad \varphi_i|_K \in P_1, \quad \forall K \in \mathcal{T}_h$

satisfying $\varphi_i(\mathbf{x}_j) = \delta_{ij}, \quad \forall i, j = 1, \dots, N$

Linear system $Au = F$ where

$$a_{ij} = \int_{\Omega} \nabla \varphi_i \cdot \nabla \varphi_j \, d\mathbf{x} + \alpha \int_{\Gamma_2} \varphi_i \varphi_j \, ds$$

$$F_i = \int_{\Omega} f \varphi_i \, d\mathbf{x} + \int_{\Gamma_1} g_1 \varphi_i \, ds + \int_{\Gamma_2} g_2 \varphi_i \, ds$$



The matrix A is SPD, sparse and banded for a proper node numbering