

Gas Physics in Cosmology: Interstellar Gas Clouds and Planetary Atmospheres

Notes from *Gravity from the ground up* Bernard Schutz, CUP 2003

Astrophysics is to a large extent a series of problems in thermodynamics, in particular the thermodynamics of gases or gas-like media such as plasma and degenerate matter. The following sections examine the gas dynamics of planetary atmospheres and stellar matter.

Gases, or at higher temperatures the gas-like phases of matter known as plasma, are the main visible constituents of our Galaxy, along with huge amounts of dust from earlier generations of stars long since burnt out as supernovae. Gas physics, chemistry and nuclear theory are all essential to understanding gas, plasma and the processes that produce dust as the basis of solar systems and planets.

We are not considering dark matter (yet) but be aware that in fact it seems to outweigh both gas and dust (including stars and planets) by a large proportion.

The early universe consisted of gaseous hydrogen, helium and lithium. The relative amounts of these materials in the visible universe is regarded as evidence in support of particular theories of cosmology. These early gas species were the sole constituents of the first stars, since stellar processes are required to build higher compounds. There were no solar systems, no planets.

The first stars and galaxies can be regarded as belonging to a very different universe to the cosmos we see today. In fact we can see the early universe by looking further back in space-time with more powerful telescopes, and yes, it looks very different.

Gases, like all matter and energy, are subject to gravity. Clouds of gas tend to collapse under their own mass, coalescing at higher densities and temperatures into spiraling or rotating disks with hot and spherical core proto-stars.

Assuming spherical symmetries in the distribution of density in a gas cloud, the net g force at any radius from the centre of the core is approximately Newton's law for mass contained inside a sphere at that radius. A gas cloud is not spherical, and this makes a great difference, but some simplifying assumptions are useful and in fact a spherical gas cloud, though unlikely to exist, would behave thermally much like any bulky gas cloud.

So assuming an average density of gas inside any sphere (the density and average density decrease with increasing radius, proportional to pressure) the mass and g-field can be calculated:

$$M = \frac{4}{3}\pi r^3 \bar{\rho}$$

or
$$g = \frac{GM}{r^2} = \frac{4}{3}\pi G r \bar{\rho}$$

Thus g falls to zero at the centre of a gas cloud. This equation has a range of solutions depending on how the density varies with radius due to temperature and pressure.

The heating of a collapsing gas is partly conservative, ie there is radiative heat loss with increasing pressure and density, but heat is retained so temperature must also increase:

$$P = \rho R_s T$$

and $\frac{P}{\rho^\gamma} = \text{constant } C$

Ideal gas law and gas constitutive law in density form.

R_s is the specific gas constant for a particular gas species; for molecular hydrogen, H_2 , $R_s = 4124 \text{ J/kgK}$. γ is related to the ratio of specific heats; $\gamma \sim 1.3$ to allow for heat loss in radiation. Schutz shows an average $\gamma = 1.357$ gives good results in a solar model.

Given initial conditions of P and ρ (or P and T , or T and ρ), the constant C is calculated and associated with subsequent temperature-pressure-density states as:

$$T = \frac{C \rho^{(\gamma-1)}}{R_s} = \frac{C^{(1/\gamma)} P^{(1-1/\gamma)}}{R_s}$$

For a given fixed mass, the density and hence temperature varies inversely with the volume. Temperature increases are thus solely dependent on the contraction of a specific mass of gas, roughly inversely proportional to the radius of the collapsing volume.

The evolution from a starting ambient temperature to a collapsing core temperature has to be modeled as a simulation in time, however it can be seen that the densities and hence gravitational fields of evolving stars are related to their temperatures.

We also need a model of the variation of pressure with depth inside a gravitational field. This can be based on the idea of hydrostatic pressure, that is pressure due to the overlying weight of matter. Allowing for variable density and gravitational acceleration, this is given by a differential equation:

$$dP = -\rho g dr$$

Example:

A LARGE, UNIFORMLY DISTRIBUTED SPHERICAL H_2 GAS CLOUD OF RADIUS 1 ly AND 1 SOLAR MASS COLLAPSES UNDER ITS OWN g .
ESTIMATE THE INITIAL PRESSURE AND TEMPERATURE.

① SAY $M = \rho_0 \frac{4\pi r^3}{3} = 1 M_\odot$
 SOLAR MASS
 $\rho_0 = \frac{3}{4\pi} \sim 0.25 \frac{M_\odot}{\text{ly}^3}$
 $= 5 \times 10^{-19} \text{ kg/m}^3 \checkmark$

② $g_0 = \frac{GM}{r^2} = \frac{7 \times 10^{-11} \times 2 \times 10^{30}}{1 \times 10^{32}} = 1.4 \times 10^{-12} \text{ m/s}^2$ @ $r = R_0$ ✓ VERY SMALL INITIAL ACCELERATION

③ $\frac{dP}{dr} = -g\rho \therefore -G \frac{4\pi r \rho^2}{3}$
 $P_s - \frac{4\pi}{6} G \rho_0^2 r^2 + P_0 = 0$ @ $r = R_0$
 $\Rightarrow P_0 = \frac{4\pi}{6} G \rho_0^2 R_0^2 = 3.5 \times 10^{-15} \text{ Pa}$

④ $T = \frac{P}{\rho R} = 1.7 \text{ K}$
 CHECK: $PV = nRT$
 $= 1.7 \times 10^{33} \times 8.3 \times 1.7 = 1.4 \times 10^{34} \text{ J} \checkmark$

$R_0 = 1 \text{ ly} = 10 \times 10^{15} \text{ m}$
 $V_0 \sim 4 \text{ ly}^3 = \frac{4\pi r^3}{3} = 4 \times 10^{48} \text{ m}^3$
 $M = M_\odot = 2 \times 10^{30} \text{ kg}$
 $= 1 \times 10^{33} \text{ moles } H_2$
 $H_2 \text{ GAS: } R = \frac{8.314}{0.002 \text{ g/mol}} = 4124 \text{ J/kgK}$

This example, simple though it is (it avoids the question of how such a uniform, isolated distribution of gas could form), shows the extremely small temperatures, pressures and densities of interstellar gas clouds.

The calculated temperature, 1.7 degrees above absolute zero, is not far from the known temperature of interstellar space, about 10 K in a typical gas cloud, and near the minimum of 2.7 K due to the background microwave radiation in intergalactic space. The temperature can be corrected by a minor adjustment of initial conditions, such as slightly smaller volume. Given the uncertainties, this simple model is not a bad estimate.

Note that the pressure and hence temperature are driven by the gravitational field *and nothing else*: there is no assumed earlier warm state or process of cooling from the Big Bang. This indicates that gravity is one among several sources of original energy.

The very low gravitational field of an interstellar gas implies that its initial collapse is very slow. The enormous distances guarantee that very long periods of time must pass for any significant collapse to occur. The process is cumulative however, and accelerates as the gas density increases. The temperature rises and the central portion begins to ionize.

At a high enough temperature the electrons of the gas become dissociated from their nuclei, creating a plasma, a mix of free nuclei and electrons. The nuclei are then free to engage in nuclear reactions, protons decaying into neutrons with the release of electrons and energy, then protons and neutrons combining to form helium nuclei, releasing more energy.

The gravitational collapse slows as internal pressures become strong enough to resist the attractive force of gravity. A stable system, a star, develops and continues to "burn" or rather *fuse*, its mass as fuel.

THE GAS CLOUD OF THE PREVIOUS EXAMPLE
COLLAPSES TO A STAR OF RADIUS $7 \times 10^6 \text{ m}$.
CALCULATE THE AVERAGE TEMPERATURE AND
PRESSURE IN THE NEW STAR.

$\Rightarrow \text{CONST.} = \frac{P}{\rho^{\gamma}} = \frac{3.5 \times 10^{-15}}{(5 \times 10^{-19})^{1.3}} = \frac{2 \times 10^9}{\text{FROM PREVIOUS EXAMPLE}} \text{ (Pa m}^3 \text{)}$

THE VOLUME OF THE STAR IS NOW

$$\frac{4\pi}{3} (700 \times 10^6)^3 = 1.4 \times 10^{27} \text{ m}^3$$

$\Rightarrow \rho = \frac{M_0}{V} = \frac{2 \times 10^{30}}{1.4 \times 10^{27}} = 1400 \text{ kg/m}^3 \checkmark$

$\Rightarrow P = \text{CONST.} \cdot \rho^{\gamma} = 2 \times 10^9 (1400)^{1.3} = 2.5 \times 10^{13} \text{ Pa}$

$\Rightarrow T = \frac{P}{\rho R} = \frac{2.5 \times 10^{13}}{1400 (1424)} = 1.2 \times 10^7 \text{ K} \checkmark$

CHECK : $P = \frac{4\pi}{3} G \rho_0^2 R_0^2 = 1.3 \times 10^{14} \text{ Pa}$
OK

The model gives a good approximation to the internal temperature of the Sun, about 10^7 K (as distinct from the surface temperature, 5800 K) and again shows a reasonable order-of-magnitude comparison of the gravitational pressure to a purely thermodynamic model. The Sun's mean pressure is around 10^{14} Pa ; the core pressure is closer to 10^{16} Pa .

The calculated average density, 1400 kg/m^3 , is the average density of the Sun. The actual density of solar material varies from between a few milligrams per cubic metre at the surface to around $150,000 \text{ kg/m}^3$ at the core. The corresponding temperatures of 5800 and 10^7 K highlight these materials as entirely unlike any substance encountered on Earth.

After billions of years of evolution, a star runs out of hydrogen and starts converting fusion products to higher and higher elements. Helium nuclei fuse to form carbon (the "triple-alpha" process); one more alpha gives oxygen. These heavier compounds build in concentric layers in the body of the star and release enormous amounts of energy.

The result is usually that the star dwindles, after a brief inflationary red giant phase; but for stars of only about 40% more than the mass of the sun a runaway fusion process occurs and the star collapses again gravitationally, imploding to a very small diameter and ejecting higher elemental fusion products in an explosion of matter: a supernova.

The matter ejected by a supernova is at first in a plasma or hot gas state, cooling to a dust of atoms, molecules and primary crystals that is very hot and energetic, traveling at high velocities outwards and cooling as it expands. The complex remnants of material propagating outwards from past supernovae are visible as *planetary nebulae*; so-called because early astronomers were unable to explain the unresolved visual detail, often with the appearance of irregularly spherical objects.

A planetary nebula is a fast-moving dust cloud, which can collide into ambient gas clouds and trigger further gravitational gas collapse, leading to the formation of new stars with crystalline planetary material in orbit. A second generation proto-star is thus a combined primary gas cloud (hydrogen, helium and trace lithium as before) with an energized dust cloud, collectively known as a *molecular cloud*. These new stars are very different to the earlier pure gas collapse stars.

Proto-stars often form in motion, as chunks of the larger molecular cloud impacted out by the residual motion of the supernova. Magnetic and gravitational forces set up a small rotation around the axis of motion, which intensifies by conservation of angular momentum as the cloud collapses.



Proto-star http://en.m.wikipedia.org/wiki/File:Hubble_sees_a_cosmic_caterpillar.jpg

Planet formation

Dust materials in a molecular cloud cool and agglomerate chemically to become icy and rocky planetessimals in orbit around the largest nearby proto-stellar mass. Chemical reactions are mediated by molecular water, forming more complex minerals in a substrate of raw elemental matter.

Gravitational forces at the scale of the dust particles are very small, mainly because they are in relatively rapid motion. Particles agglomerate at first mainly by electromagnetic mechanisms, but larger agglomerates accumulate uniform momentum in collisions and decelerate to slower velocities, so gravitational forces around them begin to develop.

The ice and dust objects of the Kuiper belt and Oort cloud on the most distant edges of the solar system are thought to be typical of this fluffy, barely agglomerated ice-rock material. Comets such as 67P (currently under exploration by the Rosetta spacecraft) are high-eccentricity samples of this material, periodically passing through the inner solar system.

As planets begin to form, temperatures in the coalescing matter again begin to rise, due partly to the absorbed energy of collisions and partly due to radioactive decay. Ices, liquids and gases melt and evaporate, becoming mobile, dragged into motion by tidal forces.

Planetessimals are initially unable to retain the free gases that emerge by out-gassing from the melt. The kinetic energy of a gas is larger than the gravitational potential of a

small mass, so gases simply evaporate from the warming planetessimals and rejoin the remnant collapsing gas cloud.

Planetary evaporation involves an understanding of molecular kinetic energy in relation to a planet's escape velocity. Given Boltzmann's law relating mean kinetic energy to temperature, the average kinetic energy of a molecule of mass m is:

$$\frac{1}{2}mV^2 = \frac{3}{2}kT$$

where k is Boltzmann's constant, $k = 1.38 \times 10^{-23} \text{ kg m}^2 \text{ s}^{-2} \text{ K}^{-1}$. If the resulting mean velocity exceeds the escape velocity, gases will tend to escape:

$$V_{\text{evap}} = \sqrt{3 \frac{k}{m} T} = \sqrt{3 R_s T} > \sqrt{2 \frac{GM}{R}}$$

Molecules have a range of velocities above and below average, so even if the temperature is well below the apparent threshold temperature, many molecules will have sufficient energy to escape. Assuming a proto-disk temperature of 600 K and a mantle density of 5000 kg/m³ the minimum planetary radius to capture molecular hydrogen is about 1600 km, or about the size of the Moon.

A new star generates a solar wind that exerts outward pressure on the gas cloud and also begins to mobilize residual dust grains. The star sweeps dust and larger grains into a harmonic series of rings, some of which break up or merge with others. Heavier planets begin to form which have the potential to retain gas as well capture other planetessimals.

Atmospheres form on these larger planets, emerging from geological sources in many different chemical forms depending on the component elements. Lighter elements such as carbon, oxygen and nitrogen as well as hydrogen and helium (both of which continue to evaporate due to their lower mass) are most common in atmospheres of methane, carbon dioxide, nitrogen-oxygen, noble gases and other more exotic molecular gases.

Hydrogen combines chemically with oxygen, releasing energy and producing enormous quantities of water. Carbon reacts with oxygen to produce carbon dioxide, or in many combinations with hydrogen to form "organic" carbon compounds. Nitrogen, hydrogen and oxygen in complex reactions with electrical and thermal energy sources, mediated by water, form amino acids. Phospholipids and proteins form cellular structures and patterns that chemically replicate themselves. Life evolves, on at least one planet that we know of.

The conditions effecting subsequent evolution of an atmosphere are very complex:

- The mix of potential gaseous elements in the planetary material;
- The strength of geologically active sources of gas;
- The gravitational and magnetic fields of the planet;
- The energy of radiation received from the star;
- The strength of the solar wind;
- The shielding of radiation and solar wind by ionised layers and the magnetic field;
- The chemistry of compounds ionised by radiation, such as ozone layers and high altitude decomposition of water vapour to oxygen and hydrogen;
- The vertical and horizontal mixing effects of the planet's rotation and daily/seasonal heating by solar energy;

- The presence of water, depending on the availability of oxygen and hydrogen and suitable reaction conditions;
- The effect of greenhouse gases such as carbon dioxide and methane;
- The effect of chemical reactions of compounds dissolved in water or the atmosphere;
- The presence of complex organic molecules capable of forming liquid and gas, such as methane and ammonia, and most importantly;
- The evolution of organisms and their metabolism of gases such as carbon dioxide, methane, molecular oxygen and nitrogen.

Schutz Chapter 7 provides an overview of comparative atmospheric physics for the terrestrial planets of the Solar System and an introductory numerical model of the Earth's atmospheric structure based on the observed thermal layers.

A theoretical model that can be calculated directly from the thermal data, ie that does not require computer simulation as per the Schutz model, is as follows:

Assuming a temperature profile such as shown in Schutz Figure 7.5 or Table 7.1, the pressure and density profile of a planet's atmosphere can be calculated vertically for z +ve upwards using the ideal gas law and static pressure equation:

$$\frac{dP}{dz} = -\rho g = -\frac{P}{R_s T} g$$

This is equivalent to:

$$\frac{dP}{P} = \frac{-g}{R_s T} dz$$

The LHS is a standard logarithmic integral, while the RHS depends on the variation of temperature with altitude. Assuming initially constant T on average,

$$\ln \left(\frac{P}{P_o} \right) = \frac{-gz}{R_s T}$$

or

$$P = P_o \exp \left(\frac{-gz}{R_s T} \right)$$

This standard exponential curve, pressure decreasing as expected with altitude, is a reasonable approximation to any of the more precise solutions. The average temperature of the Earth's atmosphere is around $T = 160$ K.

A better model assumes temperature changes linearly with altitude:

$$\frac{dP}{P} = -\frac{g}{R_s} \frac{dz}{T_o + \lambda z}$$

where λ is a lapse rate, or rate of temperature change per altitude. For the first 10 km of Earth's lower atmosphere, the lapse rate is nearly constant at -6.5 °C/km. The solution is:

$$\ln \left(\frac{P}{P_o} \right) = \frac{-g}{R_s} \left[\frac{1}{\lambda} \ln (T_o + \lambda z) \right]_0^z$$

or

$$P = P_o \left[\frac{T_o}{T_o + \lambda z} \right]^{\frac{g}{\lambda R_s}}$$

The value of λ (and the index) must include the - sign for decreasing temperature layers. The mean sea-level pressure and temperature P_o and T_o are 101,325 Pa and 188 K (15°C). The specific gas constant for air (20% oxygen, 80% nitrogen) is $R_s = 287.1$ J/kgK. This equation has been adopted as the standard Barometric Equation of meteorology.

Much of the upper atmosphere can be modeled as a series of approximately isothermal and linear thermal gradients. A solution scheme for the Earth's atmosphere uses a lapse rate of -0.0065 °C/m up to 10 km, then goes through a series of isothermal then warming, cooling and warming again layers up to 100 km. Each layer begins with a new set of initial values P_o and T_o taken as the topmost values of the previous layer.

The uppermost layer of the Earth's atmosphere increases in temperature to the hot, thin, ionized gases magnetically trapped inside the Van Allen belt and Heaviside layer. Beyond that we are in the very hot but diffuse and mobile solar atmosphere, ie the solar wind.

This model does not account for the changing gas composition, but this can be done by adjusting the value of R_s to model a varying mix of gases. The model also does not say *why* the temperature profile is this way, a more complex thermodynamic question.

Despite this complexity, atmospheric pressure decreases monotonically (ie it never rises) with altitude, to almost imperceptibly low pressures at 100 km. Since spacecraft feel almost no atmospheric drag or thermal shock at these pressures, 100 km is generally regarded as the top of the atmosphere and the beginning of "space".

Plasma Physics: Stars

A star is much like a huge and hot atmosphere that generates its own gravitational field. The equations of Newton and gas physics, used with the right choice of parameters, give reasonable approximations to nuclear and quantum astrophysics so long as we stay well away (for now) from high-curvature regions and the relativistic collapse of large stars.

We have seen how it is possible to estimate the internal conditions of a star as the end state of a polytropic compression process. With the initial mass and final radius known it is possible to solve for the internal temperature and pressure as order of magnitude estimates.

We have also seen that it is possible to model the layered structure of an atmosphere, based only on a specification of the vertical temperature profile. We can determine the temperature-pressure structure of a star in a similar way. Density measurements based on seismic observations of the solar surface provide empirical data on solar structure.

To improve on the star model we need to understand the nuclear fusion process, in particular the release of energy and the wavelengths of radiation produced, i.e. the thermodynamics of radiation. Photons have temperature, and this contributes to the internal pressure and surface colour of a star.

Subsection on stellar structure - low mass (convective), solar mass radiative-convective, large mass convective-radiative

A star's temperature at the surface is a small fraction of the internal temperature. The gas pressure at the surface is also near zero but the radiation pressure is large. The smaller surface temperature correlates with the dominant colours of stars in the visible spectrum.

Nuclear reactions in the core combine light nuclei into heavier elements by nuclear fusion. The strong and weak forces between neutrons and protons are disrupted, resulting in mass discrepancies between the input and output nuclei: the total is less than the sum of the parts. The mass difference m converts to energy in a well-known equation:

$$E = mc^2$$

where c is the speed of light, $c = 2.998 \times 10^8$ m/s.

This energy occurs in the form of electromagnetic radiation, usually at a very high frequency, typically a gamma ray or x-ray. The frequency is related to the energy by the Planck equation:

$$E = hf = h \frac{c}{\lambda}$$

where h is Planck's Constant, $h = 6.626 \times 10^{-34}$ and f, λ are the frequency and wavelength.

Lastly the temperature of a typical photon is given by Boltzmann:

$$E = \frac{3}{2} kT$$

$$T = \frac{2}{3} \frac{mc^2}{k}$$

Thus the temperature of the Sun's core relates to the loss of individual mass quanta. In the standard H-He fusion, two hydrogen nuclei, ie two protons, fuse with two neutrons to make a helium nucleus. The masses of protons and neutrons are essentially the same at $m_p = 1.67 \times 10^{-27}$ kg, but in the fusion process around 0.7% of the combining mass is lost.

So a typical mass release in H-He fusion is $m = 0.007 \times 4 \times 1.67 \times 10^{-27} = 4.6 \times 10^{-29}$ kg:

- This converts to energy as $E = 4.6 \times 10^{-29} \times (3 \times 10^8)^2 = 4.2 \times 10^{-12}$ J
- The frequency is typical of gamma radiation: $f = 4.2 \times 10^{-12} / 6.626 \times 10^{-34} = 6.3 \times 10^{21}$ Hz
- The temperature of the radiation is very high: $T = 2/3 \times 4.2 \times 10^{-12} / 1.38 \times 10^{-23} = 2 \times 10^{11}$ K

This is far higher than the temperature at the centre of the Sun, which we have already found to be around $T_c = 10^7$ K.

The discrepancy is that the radiation mixes with plasma to reach a common temperature. The radiation is *scattered* by densely packed charged nucleons and electrons. Photons engage in frequent collisions so the radiation-gas-plasma reaches thermal equilibrium at 10^7 K. The gamma radiation decreases in frequency to an average $3kT/2h = 3 \times 10^{17}$ Hz, typical of x-rays, which support a star against gravity by combined photoelectric pressure.

The densely packed core of trapped radiation is known as a *photon gas*. Like any gas, the radiation exerts a partial pressure, and contributes to the gas constant for thermal calculations. Photons are regarded as particles having momentum equivalent to the mass released in fusion, which gives a small molar mass resulting in a large gas constant: Given $m_r = 4.6 \times 10^{-29}$ kg, the specific photon gas constant $R_s = 8.31 / N_A m_r = 300,000$ J/kgK. This very large value is mixed with the smaller molecular gas constants in proportion to the photon density, which in turn depends on the rate of fusion reactions.

The ionization of hydrogen molecules into plasma protons also has a strong influence on the thermodynamics: the gas constant R_s effectively doubles from 4124 to 8248 J/kgK and the ratio of specific heats changes from $\gamma = 1.4$ (7/5) to 1.66 (5/3), with a corresponding increase in the polytropic index from $\gamma = 1.3$ to around 1.5. Gas constants and indices of different species combine in proportion to the mixture of species.

These factors are accounted for in the Standard Solar Model, which Schutz (Ch 7) approximates as a simple finite difference model with average $\gamma = 1.357$.

A consequence of scattering (usually referred to as *Compton scattering*) is that light takes a long time, up to thousands of years, to diffuse out from the core. Each scattering event lowers the energy of the radiation, the frequency decreasing in equilibrium with the gas, until it finally emerges at the characteristic colour and temperature of the Sun's surface.

Wien's law provides a formula which can be used to calculate the surface temperature of the Sun (or any star) by observation of its electromagnetic spectrum:

$$T = \frac{b}{\lambda_{peak}}$$

where λ_{peak} is the wavelength of peak radiation intensity and b is Wien's constant, $b = 2.9 \times 10^{-3}$ m.K. The Sun produces a peak blue-green wavelength $\lambda_{peak} = 5 \times 10^{-7}$ m (500 nm), which gives $T_s = 5800$ K, the solar surface temperature.

The three data points of core temperature and pressure and surface temperature enable a thermal calculation of densities at the surface and core, which we can partly correct by a knowledge of photon pressure and fusion.

The polytropic process model provides a simple relationship between pressure and temperature:

$$T = AP^\beta$$

where $\beta = 1 - 1/\gamma = 1 - 1/1.357 = 0.263$ and A can be determined from initial conditions. The average internal pressure is used as an initial condition.

i.e. for $T_c = 10^7$ K and $P = 10^{14}$, $A = 10^7(10^{14})^{-0.263} = 2080$ Pa,

Then at the surface, $T_s = 5800$ K = $2080P_s^\beta$

$$\text{so} \quad P_s = \left(\frac{5800}{2080} \right)^{1/0.263} = 50 \text{ Pa}$$

Densities at the surface and the core can be calculated using appropriate gas constants. The core gas constant is calibrated to the Standard Model Sun to give the accepted core density. The mixture of molecular hydrogen, ionized protons, helium nuclei and high energy photons suggests that the gas "constant" should be larger than 4124 J/kgK:

$$\rho_s = \frac{P_s}{R_s T_s} = \frac{50}{4124(5800)} = 2 \times 10^{-6} \text{ kg/m}^3$$

$$\rho_c = \frac{P_c}{R_c T_c} = \frac{10^{16}}{6700(10^7)} = 150 \times 10^3 \text{ kg/m}^3$$

Pressures and densities at the surface are very small compared to any natural conditions encountered on Earth, but still very large compared to the original gas cloud. Core conditions are similar orders of magnitude larger.

There is a limit to how useful this simple model is in practice. The Standard Solar Model is a subject for advanced study and would be very difficult to present here. The reader is recommended to Schutz's average model, which gives good results fairly simply.

How do we know all this?

How indeed do we know anything about the interior of the Sun? In fact we have learned to probe the Sun using seismic techniques similar to the geological exploration of the Earth's interior, by studying modes of surface vibration and analyzing their internal effects.

The Sun is a dynamic system that generates vibrations in a number of characteristic modes depending on density and gas parameters that determine the speed of sound in the solar material. The resulting surface vibrations are visible at specific frequencies.

The first observations of solar vibration were assumed to be random error, "noise" in the data. It was soon realised that they are in fact noise in the literal sense, and that this is a powerful source of information on internal conditions. See Schutz pp 99-102.

Chapter 12: Life Cycle of Stars

"Governing this [life] cycle is, as everywhere, gravity. An imbalance between gravity and heat in a transparent gas cloud leads to star formation. The long stable life of a star is a robust balance between nuclear energy generation and gravity. This balance is finally lost when the star runs out of nuclear fuel, leading to a quiet death as a white dwarf or violent death as a supernova...White dwarfs and neutron stars exist in a balance between gravity and quantum effects". Black holes represent the most extreme effect of gravity in the end state of a star, the collapse of space-time in a region of theoretically infinite curvature from which nothing, not even light, can escape.

Having established a few basic principles of the thermal physics of gas clouds and stars we can go back and add some detail to the model of star formation. We need to return to the issue avoided in the formative model of a gas cloud, namely how such a cloud could form and what conditions are needed to trigger gravitational collapse.

The pressure and gravitational forces in a molecular gas cloud, which we were able to estimate in a previous section, may be in balance such that the cloud does not collapse or may even expand. This depends on the temperature and density of the gas and involves comparing the escape velocity of the cloud to the average kinetic energy of its particles, in a manner very similar to the earlier calculation of atmosphere evaporation.

Setting the kinetic energy of a gas molecule equal to its gravitational potential energy we have:

$$\frac{1}{2}mV^2 = \frac{3}{2}kT = \frac{GMm}{r}$$

where m is the mass of the particle and M the gravitational mass of the gas cloud. The gravitational mass depends on the distribution of gas: taking again the simple model of a spherical cloud it is just the mass enclosed in a sphere of radius equal to the particle's distance from the centre of gravity:

$$M = \frac{4}{3}\pi r^3 \rho \quad PE = \frac{GMm}{r} = \frac{4}{3}\pi \rho G r^2$$

Recalling that the gravitational field is largest at the outside edge of a cloud, we can solve for R as the radius at which the gravitational potential exceeds the kinetic energy so the cloud will tend to collapse:

$$\frac{3}{2}KT = \frac{4}{3}\pi \rho GR^2$$

or

$$R = \sqrt{\frac{9kT}{8\pi\rho Gm}}$$

This analysis was improved somewhat by Sir James Jeans, who included reasoning on a more likely gas cloud shape and the effect of velocity distribution. This result is known as the *Jeans Length*, interpreted as either the minimum size of a collapsing gas cloud or the maximum size of a gravitationally stable cloud:

$$\lambda_{Jeans} = \sqrt{\frac{\pi kT}{\rho Gm}}$$

As a gas cloud collapses its density can increase out of proportion to temperature, with the result that the Jeans length decreases, making smaller regions of the same cloud subject to their own collapse. As a result, the cloud fragments into a number of separate collapsing cells on a range of scales, each of which can form a protostar. Stellar nurseries, as these regions are known, generally occur in very large molecular clouds such as the Orion Nebula, or very dense ones such as the Eagle Nebula's immense pillars of dust.

For fragmentation to occur T/ρ must decrease:

We have $P = \rho R_s T$

or $T = \frac{P}{\rho R_s}$

and $\frac{T}{\rho} = \frac{P}{\rho^2 R_s}$

but $\frac{P}{\rho^\gamma} = C \text{ constant}, \gamma < 1.4$

so $\frac{P}{\rho^2} \text{ decreases}$

It can be seen that the reduction of the Jeans length is typical, not unusual. Large gas clouds form stars on a range of scales, the largest stars forming first.

The earlier example in which the pressure at the centre of a gas cloud was calculated can now be seen as a case where the internal pressure is just equal to the gravitational pressure, leading to the smallest star that can form at a given initial temperature. The thermal pressure, a function of density and temperature, leads to a radius similar to the Jeans Length:

$$M = \frac{4}{3}\pi r^3 \rho$$

so $g = \frac{GM}{r^2} = \frac{4}{3}\pi \rho G r$

but $\frac{dP}{dr} = -g\rho = -\frac{4}{3}\pi \rho^2 G r$

i.e. $P = -\frac{4}{6}\pi \rho^2 G r^2 + P_o = 0 @ r = R$

or $P_o = \frac{4}{6}\pi \rho^2 G R^2$

This is a gravitational pressure. The actual pressure depends on density and temperature as per the ideal gas law. If the thermodynamic pressure is less than the gravitational pressure, collapse can occur:

$$P = \rho R_s T < \frac{4}{6}\pi \rho^2 G R^2$$

$$\sqrt{\frac{6 R_s T}{4 \pi \rho G}} < R$$

The specific gas constant R_s is equal to k/m , Boltzmann's constant and the molecule's mass, giving a similar estimate of the Jeans Length. The range of possible values indicates that the maximum size of a stable gas cloud is dependent on a number of factors, giving an order-of-magnitude estimate at best over a possible 3x range. For a given calculated radius of instability, the actual length could be up to 3x larger or smaller.

The end state of a collapsing gas cloud is a star with enough internal pressure to overcome the gravitational pressure. This could be a more complex problem, because it is no longer feasible to assume a constant density. However the extreme range of densities, from milligrams to tons per cubic metre, allows us to use another simplification: almost all of the mass of a star is collected near its centre. This gives a simple inverse square law for the gravitational field inside a star.

The static pressure equation can then be integrated using the process law relating pressure to density:

$$\frac{dP}{dr} = -\rho g = -\left(\frac{P}{C}\right)^{(1/\gamma)} \frac{GM}{r^2}$$

$$\frac{dP}{P^{(1/\gamma)}} = -\frac{1}{C^{1/\gamma}} \frac{GM}{r^2} dr$$

$$\frac{P^{(1-1/\gamma)}}{1-1/\gamma} = \frac{1}{C^{1/\gamma}} \frac{GM}{r} + A$$

where A is an integration constant related to the core pressure. As before we note that pressure drops to $P = 0$ at $r = R$ and hence solve for the core pressure:

$$\frac{P_c^{(1-1/\gamma)}}{1-1/\gamma} = \frac{1}{C^{1/\gamma}} \frac{GM}{R} = \frac{\rho_c}{P_c^{1/\gamma}} \frac{GM}{R}$$

$$P_c = (1 - 1/\gamma) \rho_c \frac{GM}{R}$$

Given a helioseismic estimate of the core density, $\rho_c = 150,000 \text{ kg/m}^3$, with $\gamma = 1.357$ and the known solar mass/radius we arrive at $P_c = 8 \times 10^{15} \sim 10^{16} \text{ Pa}$. The thermal pressure works out as 10^{16} Pa as before, indicating a close balance between thermal and gravitational pressure as required for the stability of the Sun and any mid-life star.

In estimating the conditions for any star other than Sol we do not have seismic data to estimate the core density, but the relationship above indicates how we can substitute around it. As Schutz (p. 95) suggests, similar stars will show a similar rate of variation in density with depth, so the ratio of core density to average density will be a constant:

$$P_c = (1 - 1/\gamma) \frac{\rho_c}{\bar{\rho}} \frac{M}{\frac{4}{3}\pi R^3} \frac{GM}{R} \propto \frac{M^2}{R^4}$$

The proportionality constant evaluates as approximately 5×10^{-10} , however the fact of the proportion is as useful as its actual value in comparative astrophysics. Schutz uses this ratio to derive a solar model. We can explore the wider consequences for stellar evolution.

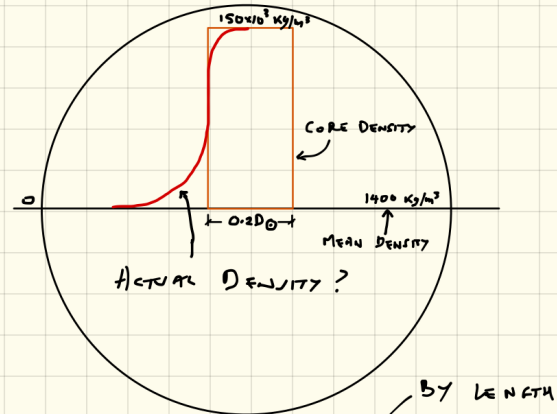
ASSUMING THE MASS OF A STAR IS COLLECTED IN ITS CORE AT THE MAXIMUM CORE DENSITY, HOW LARGE IS THE CORE?

$$M_0 = \bar{\rho} V = \rho_c V_c$$

$$V_c = M_0 / \rho_c = 2 \times 10^{30} / 150 \times 10^3 = 1.33 \times 10^{25} = \frac{4\pi}{3} r_c^3$$

$$\Rightarrow r_c = \sqrt[3]{\frac{3(1.33 \times 10^{25})}{4\pi}} = 150 \times 10^6 \text{ m}$$

$$\frac{r_c}{R_0} = \frac{150 \times 10^6}{0.7 \times 10^9} = 0.21$$

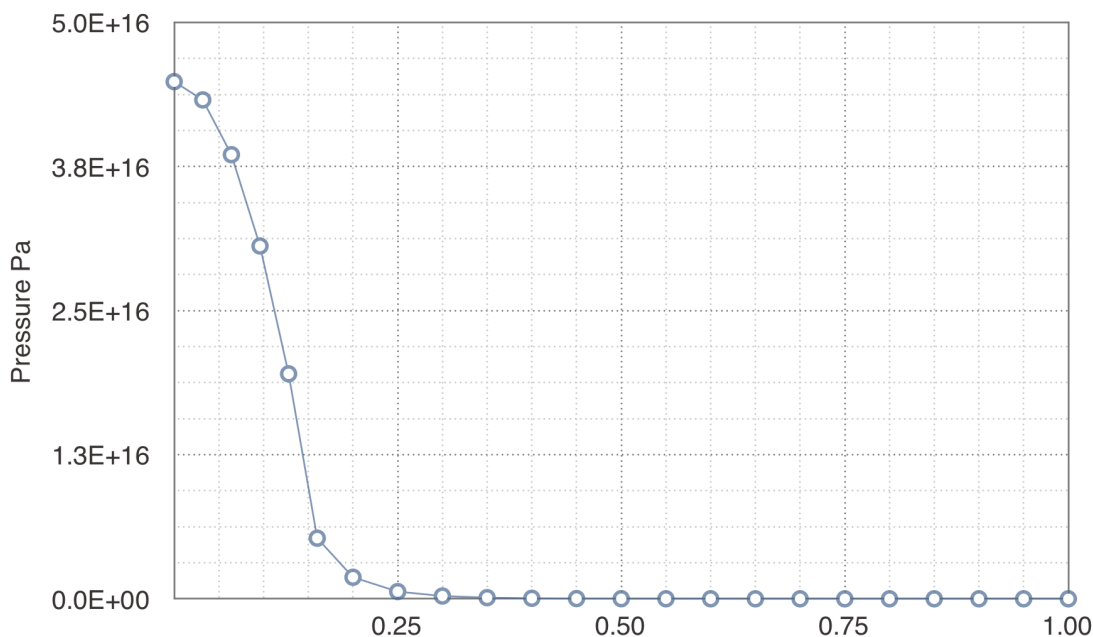


THE STAR IS 80% RELATIVELY EMPTY SPACE SURROUNDING A DENSE CORE...

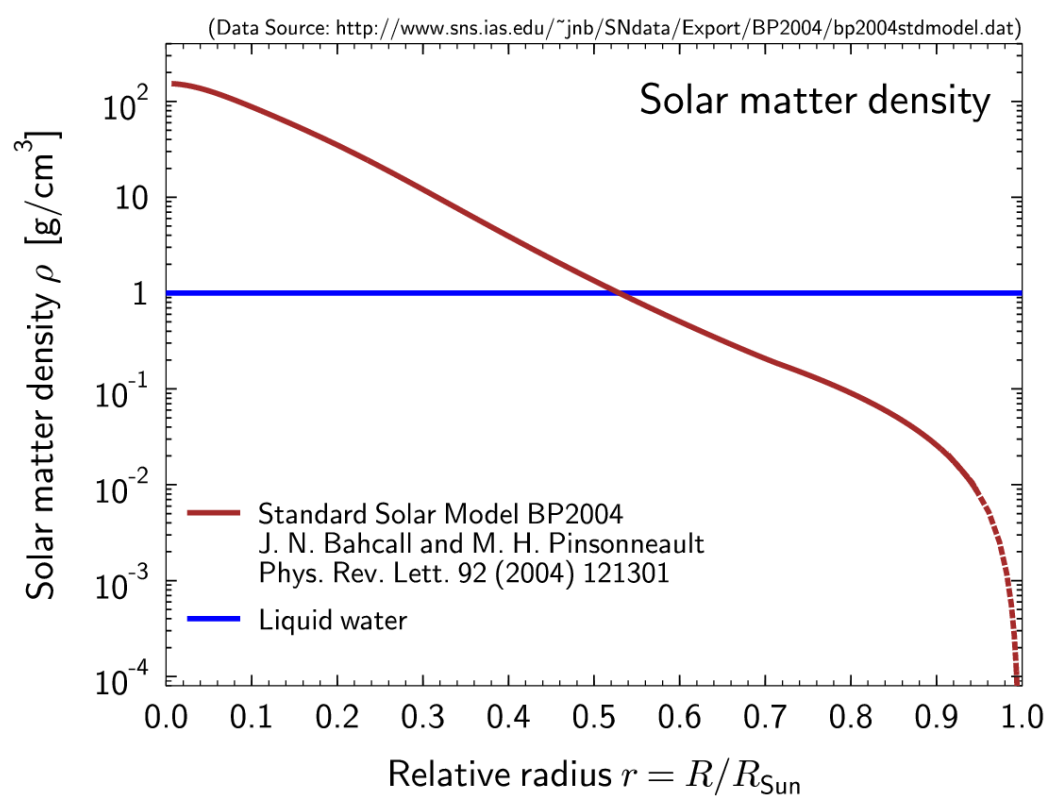
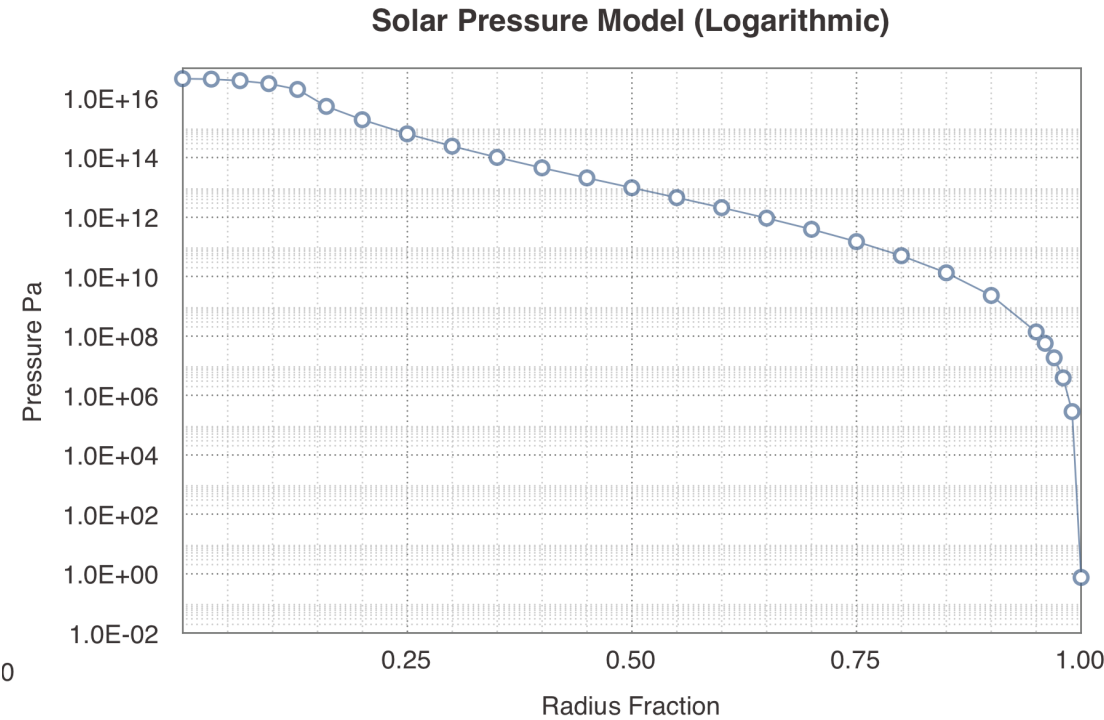
The solar model suggests that a star is relatively light through most of its depth, with an increase in density and pressure near its core. This is confirmed by the standard solar model shown in Schutz figure 8.3. The logarithmic scale of that figure indicates that the internal pressures, though orders of magnitude less than the core, are still very large.

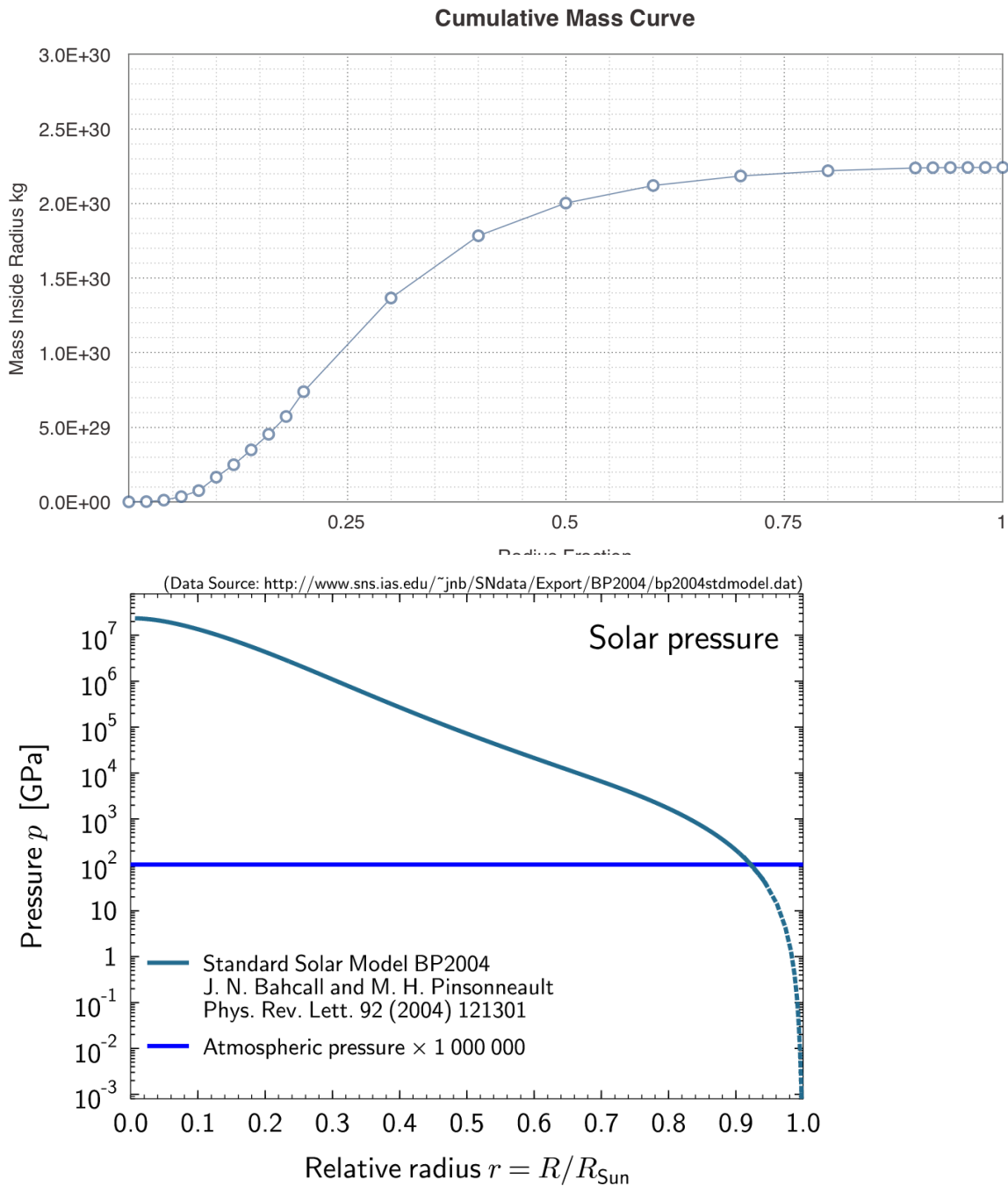
A linear-scaled two-phase solar model, with constant density in the core and constant gravitational mass controlling g in the outer layers, is shown in the figure below. This agrees with the "hat-curve" shape of the relative density profile in a star such as the sun. The large internal pressures appear equivalent to zero relative to the core on this scale.

Solar Pressure Model



The logarithmic form of this model reveals the internal pressures and agrees generally with the Standard Solar Model. It underestimates the photosphere much as Schutz's model does but captures the "shoulder" of the distribution consistent with the standard model.





Integrating the density by volume to identify the mass distribution we find that 90% of the mass is contained inside half the radius, but only 50% of the mass is within the quarter radius. Inside the core, the constant density assumption is wrong by at least an order of magnitude. This tends to explain the inaccuracies in the two-phase model.

The Standard Solar Model is easily reproduced by using the helioseismic density data (Solar matter density above) to calculate a mass curve, which determines the gravitational field at any radius, and then to apply gravity and density in a static pressure model:

$$dP = -\rho g \, dr$$

Internal temperatures are then calculated by the ideal gas law, with various assumptions about the mixture of hydrogen, helium and radiation as they determine R_s :

$$P = \rho R_s T$$